

# Realizations of countable Borel equivalence relations

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## Abstract

We study topological realizations of countable Borel equivalence relations, including realizations by continuous actions of countable groups, with additional desirable properties. Some examples include minimal realizations on any perfect Polish space, realizations as  $K_\sigma$  relations, and realizations by continuous actions on the Baire space. We also consider questions related to realizations of specific important equivalence relations, like Turing and arithmetical equivalence. We focus in particular on the problem of realization by continuous actions on compact spaces and more specifically subshifts. This leads to the study of properties of subshifts, including universality of minimal subshifts, and a characterization of amenability of a countable group in terms of subshifts. Moreover we consider a natural universal space for actions and equivalence relations and study the descriptive and topological properties in this universal space of various properties, like, e.g., compressibility, amenability or hyperfiniteness.

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# 1 Introduction

## 1.1 Topological and continuous action realizations

This paper is a contribution to the theory of countable Borel equivalence relations (CBER), a recent survey of which can be found in [Kec21b]. One of our main concerns is the subject of well-behaved, in some sense, realizations of CBER. Given CBER  $E, F$  on standard Borel spaces  $X, Y$ , resp., a Borel isomorphism of  $E$  with  $F$  is a

Borel bijection  $f: X \rightarrow Y$  which takes  $E$  to  $F$ . If such  $f$  exists, we say that  $E, F$  are **Borel isomorphic**, in symbols  $E \cong_B F$ . Generally speaking a realization of a CBER  $E$  is a CBER  $F \cong_B E$  with desirable properties.

To start with, a **topological realization** of  $E$  is an equivalence relation  $F$  on a Polish space  $Y$  such that  $E \cong_B F$ , in which case we say that  $F$  is a topological realization of  $E$  in the space  $Y$ . It is clear that every  $E$  admits a topological realization in some Polish space but we will look at topological realizations that have additional properties.

Recall here the Feldman-Moore Theorem that asserts that every CBER is induced by a Borel action of a countable group (see, e.g., [Kec21b, 2.3]). By [Kec95, 13.11] there is a Polish topology with the same Borel structure in which this action is continuous. Thus every CBER admits a topological realization in some Polish space, which is induced by a continuous action of some countable (discrete) group. We will look again at such **continuous action realizations** for which the space and the action have additional properties.

*To avoid uninteresting situations, unless it is otherwise explicitly stated or clear from the context, all the standard Borel or Polish spaces below will be uncountable and all CBER will be **aperiodic**, i.e., have infinite classes. We will denote by  $\mathcal{AE}$  the class of all aperiodic CBER on uncountable standard Borel spaces.*

Concerning topological realizations, we first show the following (in [Theorem 3.1.1](#)):

**Theorem 1.1.1.** *For every equivalence relation  $E \in \mathcal{AE}$  and every perfect Polish space  $Y$ , there is a topological realization of  $E$  in  $Y$  in which every equivalence class is dense.*

This has in particular as a consequence a stronger new version of a *marker lemma* (for the original form of the Marker Lemma see, e.g., [Kec21b, 2.15]). Let  $E$  be a CBER on a standard Borel space  $X$ . A **Lusin marker scheme** for  $E$  is a family  $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$  of Borel sets such that

- (i)  $A_\emptyset = X$ ;
- (ii)  $\{A_{sn}\}_n$  are pairwise disjoint and  $\bigsqcup_n A_{sn} \subseteq A_s$ ;
- (iii) Each  $A_s$  is a complete section for  $E$  (i.e., it meets every  $E$ -class).

We have two types of Lusin marker schemes:

(1) The Lusin marker scheme  $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$  for  $E$  is of **type I** if in (ii) above we actually have that  $\bigsqcup_n A_{sn} = A_s$  and moreover the following holds:

- (iv) For each  $x \in \mathbb{N}^{\mathbb{N}}$ ,  $\bigcap_n A_{x|n}$  is a singleton.

(Then in this case, for each  $x \in \mathbb{N}^{\mathbb{N}}$ ,  $A_n^x = A_{x|n} \setminus \bigcap_n A_{x|n}$  is a vanishing sequence of markers (i.e.,  $\bigcap_n A_n^x = \emptyset$ ).

(2) The Lusin marker scheme  $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$  for  $E$  is of **type II** if it satisfies the following:

(v) If for each  $n$ ,  $B_n = \bigsqcup \{A_s : s \in \mathbb{N}^n\}$ , then  $\{B_n\}$  is a vanishing sequence of markers.

We now have (see [Theorem 3.1.3](#)):

**Theorem 1.1.2.** *Every  $E \in \mathcal{AE}$  admits a Lusin marker scheme of type I and a Lusin marker scheme of type II.*

We next look at continuous action realizations. One such realization of  $E \in \mathcal{AE}$  would be a realization  $F$  on a compact Polish space, where  $F$  is generated by a continuous action of a countable (discrete) group. We call these **compact action realizations**. Excluding the case of smooth relations (i.e., those that admit a Borel transversal), for which such a realization is impossible, we show the following (in [Theorem 3.2.6](#)). We use the following terminology: A CBER  $E$  on  $X$  is **compressible** if there is a Borel injection  $f: X \rightarrow X$  with  $f(C) \subsetneq C$ , for every  $E$ -class  $C$ . A CBER  $E$  is **hyperfinite** if  $E = \bigcup_n E_n$ , where each  $E_n$  is a **finite** CBER (i.e., all its classes are finite) and  $E_n \subseteq E_{n+1}$ . A **minimal, compact action realization** is a compact action realization in which the group acts minimally, i.e., all the orbits are dense. Finally, for each countable group  $\Gamma$  and topological space  $X$  consider the shift action of  $\Gamma$  on  $X^\Gamma$ . The restriction of this action to a nonempty invariant closed set is called a **subshift** of  $X^\Gamma$ . We often identify a subshift with the underlying closed set.

**Theorem 1.1.3.** *Every non-smooth hyperfinite equivalence relation in  $\mathcal{AE}$  has a minimal, compact action realization. In fact this realization can be taken to be a subshift of  $2^{\mathbb{F}_2}$  if the equivalence relation is compressible and a subshift of  $2^{\mathbb{Z}}$  otherwise.*

We discuss other cases of CBER which admit such realizations in [Section 3.3](#). For each infinite countable group  $\Gamma$ , let  $E(\Gamma, 2^{\mathbb{N}})$  be the equivalence relation induced by the shift action of  $\Gamma$  on  $(2^{\mathbb{N}})^\Gamma$ . Let  $Ap((2^{\mathbb{N}})^\Gamma)$  be the **aperiodic part** of  $(2^{\mathbb{N}})^\Gamma$ , i.e., the set of points  $x$  with infinite orbit, and let  $E^{ap}(\Gamma, 2^{\mathbb{N}})$  be the restriction of  $E(\Gamma, 2^{\mathbb{N}})$  to  $Ap((2^{\mathbb{N}})^\Gamma)$ . Let also  $Fr((2^{\mathbb{N}})^\Gamma)$  be its **free part**, i.e., the set of points  $x$  such that  $\gamma \cdot x \neq x$ ,  $\forall \gamma \in \Gamma, \gamma \neq 1$ . Denote by  $F(\Gamma, 2^{\mathbb{N}})$  the restriction of  $E(\Gamma, 2^{\mathbb{N}})$  to  $Fr((2^{\mathbb{N}})^\Gamma)$ . Every aperiodic (i.e., having infinite orbits) Borel action of  $\Gamma$  on a standard Borel space is Borel isomorphic to the restriction of the shift action to an invariant Borel subset of  $Ap((2^{\mathbb{N}})^\Gamma)$  and similarly every free Borel action of  $\Gamma$  on a standard Borel space is Borel isomorphic to the restriction of the shift action to an invariant Borel subset of  $Fr((2^{\mathbb{N}})^\Gamma)$ .

Also a CBER is **universal** if every CBER can be Borel reduced to it. As opposed to [Theorem 1.1.3](#), the next results (see [Theorem 3.3.1](#) and [Corollary 3.6.6](#)) show that some very complex CBER have compact action realizations.

**Theorem 1.1.4.** *(i) For every infinite countable group  $\Gamma$ ,  $F(\Gamma, 2^{\mathbb{N}})$  admits a compact action realization. If  $\Gamma$  is also finitely generated, then  $E^{ap}(\Gamma, 2^{\mathbb{N}})$  admits a compact action realization. In fact in both cases such a realization can be taken to be a subshift of  $(2^{\mathbb{N}})^{\Gamma}$ .*

*(ii) Every compressible, universal CBER admits a compact action realization. In fact such a realization can be taken to be a subshift of  $2^{\mathbb{F}_4}$ .*

In particular, it follows that arithmetical equivalence  $\equiv_A$  on  $2^{\mathbb{N}}$  has a compact action realization but it is unknown if Turing equivalence  $\equiv_T$  has such a realization. More generally, we do not know whether *every* non-smooth CBER has a compact action realization. We also do not know if *every* non-smooth CBER even admits some other kinds of realizations, for example transitive (i.e., having at least one dense orbit) continuous action realizations on arbitrary or special types of Polish spaces. These problems as well as the situation with smooth CBER in such realizations are discussed in [Section 3.2](#).

Returning to Turing equivalence, in [Section 3.4](#), we discuss topological realizations of Turing equivalence  $\equiv_T$  and show that it admits a Baire class 2 isomorphism to an equivalence relation given by a continuous group action on the Baire space  $\mathbb{N}^{\mathbb{N}}$ . We do not know if this can be improved to Baire class 1 but we also show that no such isomorphism can be below the identity on a cone of Turing degrees.

In [Section 3.5](#) we discuss some special properties of continuous actions of countable groups on compact Polish spaces, related to compressibility and paradoxical decompositions, that may be relevant to compact action realizations.

## 1.2 Subshifts

Related to [Theorem 1.1.4](#), we call a countable group  $\Gamma$  **minimal subshift universal** if there is a minimal subshift of  $2^{\Gamma}$  on which the restriction of the shift equivalence relation is universal. Then we have, see [Corollary 3.6.5](#) and [Corollary 3.6.6](#):

**Theorem 1.2.1.** *Let  $\Gamma$  and  $\Lambda$  be infinite groups, where  $\Lambda$  admits a Borel action on a standard Borel space whose induced equivalence relation is universal (e.g., any group containing  $\mathbb{F}_2$ ). Then the wreath product  $\Gamma \wr \Lambda$  is minimal subshift universal. In particular,  $\mathbb{F}_3$  is minimal subshift universal.*

We do not know if  $\mathbb{F}_2$  is minimal subshift universal.

It is well known that a countable group  $\Gamma$  is **amenable** iff every continuous action of  $\Gamma$  on a compact space admits an invariant Borel probability measure. Call a class  $\mathcal{F}$  of such actions a **test for amenability** for  $\Gamma$  if  $\Gamma$  is amenable provided that every action in  $\mathcal{F}$  admits an invariant Borel probability measure. In [GdlH97] it is shown that the class of actions on  $2^{\mathbb{N}}$  is a test for amenability for any group. Equivalently this says that the class of all subshifts of  $(2^{\mathbb{N}})^{\Gamma}$  is a test of amenability for  $\Gamma$ . It turns out that the strongest result along these lines is actually true, namely that the class of all subshifts of  $2^{\Gamma}$  is a test of amenability for  $\Gamma$ , see [Theorem 3.7.1](#). This gives another characterization of amenability.

**Theorem 1.2.2.** *Let  $\Gamma$  be a countable group. Then  $\Gamma$  is amenable iff every subshift of  $2^{\Gamma}$  admits an invariant Borel probability measure.*

Our proof of [Theorem 1.2.2](#) is based on an explicit construction of a subshift of  $2^{\Gamma}$  with no invariant Borel probability measure for every non-amenable group  $\Gamma$ . Andy Zucker communicated subsequently to the authors a different proof based on ideas of abstract topological dynamics, especially the concept of strongly proximal flows.

We study in [Section 3.8](#) a universal space for actions and equivalence relations and the descriptive or topological properties of various subclasses.

Fix a countable group  $\Gamma$ . For any Polish space  $X$ , define the standard Borel space of subshifts of  $X^{\Gamma}$  as follows:

$$\text{Sh}(\Gamma, X) = \{F \in F(X^{\Gamma}) : F \text{ is } \Gamma\text{-invariant}\}$$

If  $X$  is compact, we view this as a compact Polish space with the Vietoris topology.

Consider the Hilbert cube  $\mathbb{I}^{\mathbb{N}}$ . Every compact Polish space is (up to homeomorphism) a closed subspace of  $\mathbb{I}^{\mathbb{N}}$ , and thus every  **$\Gamma$ -flow** (i.e., a continuous action of  $\Gamma$  on a compact Polish space) is (topologically) isomorphic to a subshift of  $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$ . We can thus consider the compact Polish space  $\text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$  as the universal space of  $\Gamma$ -flows.

Similarly consider the product space  $\mathbb{R}^{\mathbb{N}}$ . Every Polish space is (up to homeomorphism) a closed subspace of  $\mathbb{R}^{\mathbb{N}}$ , and thus every continuous  $\Gamma$ -action on a Polish space is (topologically) isomorphic to a subshift of  $(\mathbb{R}^{\mathbb{N}})^{\Gamma}$ . We can thus consider the standard Borel space  $\text{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$  as the universal space of continuous  $\Gamma$ -actions.

In particular taking  $\Gamma = \mathbb{F}_{\infty}$ , the free group with a countably infinite set of generators, we see that every CBER is Borel isomorphic to the equivalence relation  $E_F$  induced on some subshift  $F$  of  $(\mathbb{R}^{\mathbb{N}})^{\mathbb{F}_{\infty}}$  and so we can view  $\text{Sh}(\mathbb{F}_{\infty}, \mathbb{R}^{\mathbb{N}})$  also as the universal space of CBER and study the complexity of various classes of CBER (like, e.g., smooth, aperiodic, hyperfinite, etc.) as subsets of this universal space. Similarly we can view  $\text{Sh}(\mathbb{F}_{\infty}, \mathbb{I}^{\mathbb{N}})$  as the universal space of CBER that admit a compact action

realization. In this case we can also consider complexity questions as well as generic questions of various classes.

Let  $\Phi$  be a property of continuous  $\Gamma$ -actions on Polish spaces which is invariant under (topological) isomorphism. Let

$$\text{Sh}_\Phi(\Gamma, X) = \{F \in \text{Sh}(\Gamma, X) : F \models \Phi\},$$

where we write  $F \models \Phi$  to mean that  $F$  has the property  $\Phi$ .

We will consider below the following  $\Phi$ , where for the definition of the concepts in 7)–10) below see [Section 3.8](#), **(B)**.

- 1) fin: finite equivalence relation;
- 2) sm: smooth equivalence relation;
- 3) free: free action;
- 4) aper: aperiodic equivalence relation;
- 5) comp: compressible equivalence relation;
- 6) hyp: hyperfinite equivalence relation;
- 7) amen: amenable equivalence relation;
- 8) measHyp: measure-hyperfinite equivalence relation
- 9) freeMeasHyp: free action + measure-hyperfinite equivalence relation;
- 10) measAmen: measure-amenable action.

We summarize in the following table what we can prove concerning the descriptive or generic properties of the  $\Phi$  above.

| $\Phi$      | $\text{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$ | $\text{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N})$ |                              |
|-------------|---|---|------------------------------|
| fin         | meager  | $\mathbf{\Pi}_1^1$ -complete                    |                              |
| sm          |   |   |                              |
| free        | comeager  | $G_\delta$                                      | $\mathbf{\Pi}_1^1$ -complete |
| aper        |   |   |                              |
| comp        |   | open  |                              |
| hyp         | ?   | $\Sigma_2^1, \mathbf{\Pi}_1^1$ -hard            |                              |
| amen        |   |   |                              |
| measHyp     | comeager  | $\mathbf{\Pi}_1^1$ -complete                    |                              |
| freeMeasHyp |   | $G_\delta$                                      | $\mathbf{\Pi}_1^1$ -complete |
| measAmen    |   |   |                              |

In this table,  $\Gamma$  is an infinite group,  $\Gamma$  is residually finite in the “ $\mathbf{\Pi}_1^1$ -complete” entry of the first two rows,  $\Gamma$  is non-amenable in the “comeager” entry of the fifth row,  $\Gamma$  is non-amenable and residually finite in the “ $\mathbf{\Pi}_1^1$ -hard” and “ $\mathbf{\Pi}_1^1$ -complete” entries of the last five rows, and  $\Gamma$  is exact in the “comeager” entry of the last four rows (where a group is **exact** if it admits an amenable action on a compact Polish space; see [BO08, Chapter 5]). We do not know if hyperfiniteness is generic in  $\text{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$  for every infinite  $\Gamma$  (or just  $\mathbb{F}_2$ ) and we do not know the exact descriptive complexity of hyperfiniteness.

### 1.3 $K_\sigma$ realizations

Clinton Conley also raised the question of whether every  $E \in \mathcal{AE}$  admits a  $K_\sigma$  realization in a Polish space. We show in [Theorem 3.9.1](#) that one can even obtain a transitive  $K_\sigma$  realization on  $2^\mathbb{N}$ , where an equivalence relation is **transitive** if it has at least one dense class. This raises the related question of whether every  $E \in \mathcal{AE}$  admits a minimal  $K_\sigma$  (or even  $F_\sigma$ ) realization in a Polish space, where an equivalence relation is called **minimal** if all its classes are dense. In view of [Theorem 1.1.3](#), every non-smooth hyperfinite equivalence relation in  $\mathcal{AE}$  has a minimal  $K_\sigma$  realization on a compact Polish space and Solecki in [Sol02] has shown that this fails for smooth relations, but this is basically the extent of our knowledge in this matter. Call a CBER on a compact Polish space  $X$  **compactly graphable** if there is a compact graphing of  $E$ , i.e., a compact graph (irreflexive, symmetric relation)  $K \subseteq E$  so that the  $E$ -classes are the connected components of  $K$ . Clearly every such  $E$  is  $K_\sigma$ . We also show in [Theorem 3.9.5](#) that every hyperfinite and every compressible CBER in  $\mathcal{AE}$  has a **compactly graphable realization**, i.e., is Borel isomorphic to an



equivalence relation on a compact Polish space that is compactly graphable. We do not know if this is true for every  $E \in \mathcal{AE}$ . Finally in [Section 3.10](#) we study a  $\sigma$ -ideal associated with a  $K_\sigma$  CBER.

## 1.4 The Borel inclusion order

In connection with these realization problems, we were also led to consider the following quasi-order on CBER, which we call the **Borel inclusion order**. Given CBER  $E, F$  on standard Borel spaces, we put  $E \subseteq_B F$  if there is  $E' \cong_B E$  with  $E' \subseteq F$ .

*Below, unless otherwise explicitly stated or understood from the context, by a **measure** on a standard Borel space we will always mean a Borel probability measure.*

For each CBER  $E$ , we denote by  $\text{EINV}_E$  the set of ergodic, invariant measures for  $E$  and by  $|\text{EINV}_E| \in \{0, 1, 2, \dots, \aleph_0, 2^{\aleph_0}\}$  its cardinality.

Recall here Nadkarni's Theorem (see, e.g., [[Kec21b](#), 4.C]) which asserts that for a CBER  $E$  the following are equivalent:

- (i)  $E$  has no invariant measure;
- (ii)  $|\text{EINV}_E| = 0$  ;
- (iii)  $E$  is compressible.

We now have the following result (see [Proposition 2.1.3](#), [Theorem 2.2.3](#) and [Corollary 2.2.6](#)), where  $\mathcal{AH}$  is the class of hyperfinite relations in  $\mathcal{AE}$ .

**Theorem 1.4.1.** *(i) If  $E \subseteq_B F$  are in  $\mathcal{AE}$ , then  $|\text{EINV}_E| \geq |\text{EINV}_F|$  and if  $E, F \in \mathcal{AH}$ , then  $E \subseteq_B F \iff |\text{EINV}_E| \geq |\text{EINV}_F|$ .*

*(ii) For any  $E \in \mathcal{AE}$ , there is  $F \in \mathcal{AH}$  with  $F \subseteq E$  such that moreover  $\text{EINV}_E = \text{EINV}_F$ .*

Using this and the classification theorem for hyperfinite CBER from [[DJK94](#), 9.1], one can then prove the next result (see [Theorem 2.2.5](#) and [Proposition 2.3.1](#)), where we use the following terminology and notation:

For each CBER  $E$  and standard Borel space  $S$ ,  $SE$  is the direct sum of “ $S$ ” copies of  $E$  (see [Section 2.1](#)). We let  $E_0$  be the equivalence relation on  $2^{\mathbb{N}}$  given by  $x E_0 y \iff \exists m \forall n \geq m (x_n = y_n)$ ;  $E_t$  is the equivalence relation on  $2^{\mathbb{N}}$  given by  $x E_t y \iff \exists m \exists n \forall k (x_{m+k} = y_{n+k})$ ;  $I_{\mathbb{N}} = \mathbb{N}^2$ ;  $E_\infty$  is a universal under Borel embeddability CBER; and  $E \times F$  is the product of  $E$  and  $F$ . Finally  $\subset_B$  is the strict part of  $\subseteq_B$  and for any quasi-order  $\preceq$  with strict part  $\prec$  on a set  $Q$  and  $q, r \in Q$ , we say that  $r$  is a **successor** to  $q$  if  $q \prec r$  and  $(s \prec r \implies s \preceq q)$ . Finally, for each cardinal  $\kappa \in \{0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$ , let  $\mathcal{AE}_\kappa$  be the class of all  $E \in \mathcal{AE}$  such that  $|\text{EINV}_E| = \kappa$ . Thus by Nadkarni's Theorem  $\mathcal{AE}_0$  is the class of compressible relations. We also let for  $\kappa > 0$ ,  $\kappa E = SE$ , where  $S$  is a standard Borel space of cardinality  $\kappa$ .

**Theorem 1.4.2.** (i)  $\mathbb{R}E_0 \subseteq_B \mathbb{N}E_0 \subseteq_B \cdots \subseteq_B 3E_0 \subseteq_B 2E_0 \subseteq_B E_0 \subseteq_B E_t$ , each equivalence relation in this list is a successor in  $\subseteq_B$  of the one preceding it and  $\mathbb{N}E_0$  is the infimum in  $\subseteq_B$  of the  $nE_0, n \in \mathbb{N} \setminus \{0\}$ .

(ii)  $\mathbb{R}I_{\mathbb{N}} \subseteq_B E_t$  and  $E_t$  is a successor of  $\mathbb{R}I_{\mathbb{N}}$  in  $\subseteq_B$ .

(iii)  $\mathbb{R}I_{\mathbb{N}}$  is  $\subseteq_B$ -minimum in  $\mathcal{AE}_0$  and  $E_t$  is  $\subseteq_B$ -minimum among the non-smooth elements of  $\mathcal{AE}_0$ . (B. Miller) Also  $E_{\infty} \times I_{\mathbb{N}}$  is  $\subseteq_B$ -maximum in  $\mathcal{AE}_0$ .

(iv) For each  $\kappa > 0$ ,  $\kappa E_0$  is a  $\subseteq_B$ -minimum element of  $\mathcal{AE}_{\kappa}$  but  $\mathcal{AE}_{\kappa}$  has no  $\subseteq_B$ -maximum element.

(v) Let  $\kappa \leq \lambda$ . Then for every  $E \in \mathcal{AE}_{\lambda}$ , there is  $F \in \mathcal{AE}_{\kappa}$  such that  $E \subseteq_B F$ .

In particular  $\mathbb{R}E_0$  is  $\subseteq_B$ -minimum non-smooth in  $\mathcal{AE}$  and  $E_{\infty} \times I_{\mathbb{N}}$  is  $\subseteq_B$ -maximum in  $\mathcal{AE}$ . Thus one has the following version of the Glimm-Effros Dichotomy for  $\subseteq_B$  (see [Corollary 2.2.7](#)):

**Theorem 1.4.3.** Let  $E \in \mathcal{AE}$ . Then exactly one of the following holds:

(i)  $E$  is smooth,

(ii)  $\mathbb{R}E_0 \subseteq_B E$ .

## 1.5 2-adequate groups

For each infinite countable group  $\Gamma$  and standard Borel space  $X$  consider the shift action of  $\Gamma$  on  $X^{\Gamma}$  and let  $E(\Gamma, X)$  be the associated equivalence relation and  $E^{ap}(\Gamma, X)$  be its aperiodic part, i.e., the restriction of  $E(\Gamma, X)$  to the set of points with infinite orbits. Consider now a Borel action of  $\Gamma$  on an uncountable standard Borel space, which we can assume is equal to  $\mathbb{R}$ . Then the map  $f: X \rightarrow \mathbb{R}^{\Gamma}$  given by  $x \mapsto p_x$ , where  $p_x(\gamma) = \gamma^{-1} \cdot x$ , is an equivariant Borel embedding of this action to the shift action on  $\mathbb{R}^{\Gamma}$ . Thus every aperiodic CBER  $E$  induced by a Borel action of  $\Gamma$  can be realized as (i.e., is Borel isomorphic to) the restriction of  $E^{ap}(\Gamma, \mathbb{R})$  to an invariant Borel set. By a result in [\[JKL02, 5.5\]](#) we also have  $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, \mathbb{N})$ , so such realizations exist for  $E^{ap}(\Gamma, \mathbb{N})$  as well. We consider here the question of whether these realizations can be achieved in the optimal form, i.e., replacing  $E^{ap}(\Gamma, \mathbb{N})$  by  $E^{ap}(\Gamma, 2)$ . This is equivalent to the statement that  $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2)$ . If this happens then we call the group  $\Gamma$  **2-adequate**.

Using a recent result of Hochman-Seward, we show the following (see [Theorem 4.0.4](#)):

**Theorem 1.5.1.** Every infinite countable amenable group is 2-adequate.

This in particular answers in the negative a question of Thomas [\[Tho12, Page 391\]](#), who asked whether there are infinite countable amenable groups  $\Gamma$  for which  $E(\Gamma, \mathbb{R})$  is not Borel reducible to  $E(\Gamma, 2)$ .

We also show the following (see [Corollary 4.0.9](#) and [Proposition 4.0.11](#)):

**Theorem 1.5.2.** (i) *The free product of any countable group with a group that has an infinite amenable factor and thus, in particular, the free groups  $\mathbb{F}_n, 1 \leq n \leq \infty$ , are 2-adequate.*

(ii) *Let  $\Gamma$  be  $n$ -generated,  $1 \leq n \leq \infty$ . Then  $\Gamma \times \mathbb{F}_n$  is 2-adequate. In particular, all products  $\mathbb{F}_m \times \mathbb{F}_n, 1 \leq m, n \leq \infty$ , are 2-adequate.*

On the other hand there are groups which are not 2-adequate (see [Theorem 4.0.12](#)).

**Theorem 1.5.3.** *The group  $\mathrm{SL}_3(\mathbb{Z})$  is not 2-adequate.*

We do not know if there is a characterization of 2-adequate groups.

## 1.6 Some other classes of groups

In the course of the previous investigations two other classes of groups have been considered. A countable group  $\Gamma$  is called **hyperfinite generating** if for every  $E \in \mathcal{AH}$  there is a Borel action of  $\Gamma$  that generates  $E$ . We provide equivalent formulations of this property in [Proposition 5.1.1](#) and show in [Corollary 5.1.2](#) that all countable groups with an infinite amenable factor are hyperfinite generating, while no infinite countable group with property (T) has this property (see [Proposition 5.1.3](#)).

Finally we say that an infinite countable group  $\Gamma$  is **dynamically compressible** if every  $E \in \mathcal{AE}$  generated by a Borel action of  $\Gamma$  can be Borel reduced to a compressible  $F \in \mathcal{AE}$  induced by a Borel action of  $\Gamma$ . We show in [Proposition 5.2.3](#) that every infinite countable amenable group is dynamically compressible and the same is true for any countable group that contains a non-abelian free group (see [Proposition 5.2.4](#)). However there are infinite countable groups that fail to satisfy these two conditions but they are still dynamically compressible (see [Proposition 5.2.5](#)). We do not know if *every* infinite countable group is dynamically compressible.

## 1.7 Organization

The paper is organized as follows. In Section 2, we study the structure of the Borel inclusion order on countable Borel equivalence relations. In Section 3, we consider topological realizations of countable Borel equivalence relations. In Section 4, we introduce and study the concept of 2-adequate groups, and in Section 5 we discuss results concerning the concepts of hyperfinite generating groups and dynamically compressible groups. In Section 6, we collect some of the main open problems discussed in this paper. Finally in [Appendix A](#) we discuss various notions of amenability for actions of countable groups that are relevant to the results in [Section 3.8](#).

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## 2 The Borel inclusion order of countable Borel equivalence relations

### 2.1 General properties

**Definition 2.1.1.** Let  $E, F$  be CBER on standard Borel spaces  $X, Y$ , resp. We put  $E \subseteq_B F$  if there is a Borel isomorphism  $f: X \rightarrow Y$  with  $f(E) \subseteq F$ .

It is clear that  $\subseteq_B$  is a quasi-order on CBER, which we call the **Borel inclusion order**. We also let  $E \subset_B F \iff E \subseteq_B F \ \& \ F \not\subseteq_B E$  be the strict part of this order.

Recall that a **homomorphism** of an equivalence relation  $E$  on  $X$  to an equivalence relation  $F$  on  $Y$  is a map  $f: X \rightarrow Y$  such that  $xEy \implies f(x)Ff(y)$ . Thus  $E \subseteq_B F$  iff there is a bijective Borel homomorphism of  $E$  to  $F$ .

We will study in this section the structure of this inclusion order on aperiodic CBER in uncountable standard Borel spaces.

We first prove some basic facts concerning the Borel inclusion order that will be repeatedly used in the sequel. Recall that a CBER  $E$  on  $X$  is **smooth** if it admits a Borel selector and **compressible** if there is Borel injection  $f: X \rightarrow X$  such that for each  $E$ -class  $C$ ,  $f(C) \subsetneq C$ . We also let  $I_{\mathbb{N}}$  be the equivalence relation  $\mathbb{N}^2$  on  $\mathbb{N}$  and for each equivalence relation  $E$  on  $X$  and standard Borel space  $S$ , we let  $SE$  be the direct sum of “ $S$ ” copies of  $E$ , i.e., the equivalence relation on  $S \times X$  defined by  $(s, x)SE(t, y) \iff s = t \ \& \ xEy$ . It is clear that there is a unique up to Borel isomorphism (which we denote by  $\cong_B$ ), smooth aperiodic CBER, namely  $\mathbb{R}I_{\mathbb{N}}$ .

**Proposition 2.1.2.** (i) If  $E \subseteq_B F$  and  $F$  is smooth, then  $E$  is smooth.

(ii)  $E$  is compressible iff  $\mathbb{R}I_{\mathbb{N}} \subseteq_B E$ . Therefore if  $E \subseteq_B F$  and  $E$  is compressible, then  $F$  is compressible.

*Proof.* (i) By the Feldman-Moore Theorem (see, e.g., [Kec21b, 2.B]), there is a Borel action of a countable group  $\Gamma = \{\gamma_n\}$  on  $X$  (the space of  $F$ ) which induces  $F$ , i.e.,  $xFy \iff \exists \gamma \in \Gamma(\gamma \cdot x = y)$ . Let  $f$  be a Borel selector for  $F$  and define for each

$x \in X$ ,  $n(x) =$  the least  $n$  with  $\gamma_n \cdot f(x)Ex$ . Then  $g(x) = \gamma_n \cdot f(x)$  is a Borel selector for  $E$ .

(ii) This follows from [Kec21b, 2.23]. □

The number of ergodic, invariant probability Borel measures for a CBER  $E$  will play an important role in the sequel. We denote by  $\text{EINV}_E$  the set of ergodic, invariant probability Borel measures and by  $|\text{EINV}_E|$  its cardinality. Since  $\text{EINV}_E$  can be viewed in a canonical way as a standard Borel space (see, e.g., [Kec21b, 4.10]) we have that  $|\text{EINV}_E| \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ . Moreover by Nadkarni's Theorem, see [Kec21b, 4.C], we have that  $|\text{EINV}_E| = 0$  iff  $E$  is compressible.

We note here the following basic fact:

**Proposition 2.1.3.** *If  $E \subseteq_B F$ , then  $|\text{EINV}_E| \geq |\text{EINV}_F|$ .*

*Proof.* This is clear when  $|\text{EINV}_F| = 0$ . Otherwise assume that  $E \subseteq F$  live on a space  $X$  and  $F$  admits at least one invariant measure. Consider then the ergodic decomposition  $\{X_e\}_{e \in \text{EINV}_F}$  of  $F$ , see [Kec21b, 4.11]. Then for each  $e \in \text{EINV}_F$ ,  $X_e$  is  $E$ -invariant and  $e$  is an invariant measure for  $E|_{X_e}$ , thus  $X_e$  supports at least one ergodic, invariant measure for  $E$ , say  $e'$ . Since the map  $e \mapsto e'$  is injective the proof is complete. □

We will next show that many subclasses of  $\mathcal{AE}$ , including  $\mathcal{AE}$  itself, admit maximum under  $\subseteq_B$  elements. This was proved for  $\mathcal{AE}$  by Ben Miller, see [Kec21b, 11.E], and the proof below is an adaptation of his argument to a more general context. Later we will show the existence of a minimum under  $\subseteq_B$  non-smooth element of  $\mathcal{AE}$  (see the paragraph following Corollary 2.2.7).

Below for equivalence relations  $E, F$  on spaces  $X, Y$ , resp., we let  $E \sqsubseteq_B F$  iff there is a Borel injection  $f: X \rightarrow Y$  such that  $xEy \iff f(x)Ff(y)$ . Again  $\sqsubseteq_B$  is a quasi-order on CBER. Also we let  $E \times F$  be the equivalence relation on  $X \times Y$  given by  $(x, y)E \times F(x', y') \iff (xE'x' \ \& \ yF'y')$ . We now have:

**Theorem 2.1.4.** *Let  $\mathcal{E} \subseteq \mathcal{AE}$  be a class of CBER such that  $\mathcal{E}$  contains a maximum under  $\sqsubseteq_B$  element  $E$  such that  $E \times I_{\mathbb{N}} \in \mathcal{E}$ . Then  $E \times I_{\mathbb{N}} \in \mathcal{E}$  is  $\sqsubseteq_B$ -maximum for  $\mathcal{E}$ .*

*Proof.* We start with the following fact, where for two equivalence relations  $F, G$ ,  $F \oplus G$  is their **direct sum**.

**Lemma 2.1.5.** *Let  $R$  be compressible. Then for any  $S \in \mathcal{AE}$ ,  $S \subseteq_B R \oplus S$ .*

*Proof.* Suppose  $S$  lives on the space  $X$ . Then there is an  $S$ -invariant Borel set  $X_0 \subseteq X$  such that  $S|_{X_0} \cong_B \mathbb{R}I_{\mathbb{N}}$ . Since  $\mathbb{R}I_{\mathbb{N}} \oplus \mathbb{R}I_{\mathbb{N}} \cong_B \mathbb{R}I_{\mathbb{N}}$ , we have, by Proposition 2.1.2, that  $S \cong_B \mathbb{R}I_{\mathbb{N}} \oplus S \subseteq_B R \oplus S$ . □

Let now  $F \in \mathcal{E}$  in order to show that  $F \subseteq_B E \times I_{\mathbb{N}}$ . Since  $F \subseteq_B E$ , there is  $G$  such that  $F \oplus G \subseteq_B E$ . Recalling (see, e.g., [Kec21b, 2.23]) that for any CBER  $R$ ,  $R \times I_{\mathbb{N}}$  is compressible, we have, by Lemma 2.1.5, that  $F \subseteq_B F \oplus (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}})$ . Note now that  $F \oplus (F \times I_{\mathbb{N}}) \subseteq_B F \times I_{\mathbb{N}}$ , therefore  $F \subseteq_B F \oplus (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}}) \subseteq_B (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}}) \cong_B (F \oplus G) \times I_{\mathbb{N}} \subseteq_B E \times I_{\mathbb{N}}$ .  $\square$

In particular this applies to the following classes  $\mathcal{E}$ : hyperfinite,  $\alpha$ -amenable (see [Kec21b, 8.B]), treeable,  $\mathcal{AE}$ .

## 2.2 Hyperfiniteness

We will discuss here the inclusion order on the hyperfinite equivalence relations. Recall first the following well-known fact (see, e.g., [Kec21b, 7.13]):

**Proposition 2.2.1.** *If  $E$  is hyperfinite and  $F \subseteq_B E$ , then  $F$  is hyperfinite.*

Thus the class  $\mathcal{AH}$  of hyperfinite aperiodic CBER forms an initial segment in  $\subseteq_B$ . It is also downwards cofinal in  $\subseteq_B$  in view of the following standard result (see, e.g., [Kec21b, 2.10]):

**Theorem 2.2.2.** *For any  $E \in \mathcal{AE}$ , there is  $F \in \mathcal{AH}$  with  $F \subseteq E$ .*

We will actually need a more precise version of this result, see [Kec21b, 7.12]. Since a proof of this result has not appeared in print before, we will include it below.

**Theorem 2.2.3.** *For any  $E \in \mathcal{AE}$ , there is  $F \in \mathcal{AH}$  with  $F \subseteq E$  such that moreover  $\text{EINV}_E = \text{EINV}_F$ .*

*Proof.* We will need the following lemma. Below  $E_0$  is the equivalence relation on  $2^{\mathbb{N}}$  defined by  $x E_0 y \iff \exists m \forall n \geq m (x_n = y_n)$  and  $\mu_0$  is the product measure on  $2^{\mathbb{N}}$ , where  $2 = \{0, 1\}$  is given the uniform  $(\frac{1}{2}, \frac{1}{2})$  measure. Then  $\mu_0$  is the unique element of  $\text{EINV}_{E_0}$ .

**Lemma 2.2.4.** *Let  $E$  be a CBER on a standard Borel space  $X$  and let  $\mu \in \text{EINV}_E$ . Then there is an  $E$ -invariant Borel set  $X_0 \subseteq X$  with  $\mu(X_0) = 1$ , an  $E_0$ -invariant Borel set  $C_0 \subseteq 2^{\mathbb{N}}$  with  $\mu_0(C_0) = 1$  and a Borel isomorphism  $f: C_0 \rightarrow X_0$  such that  $f_*\mu_0 = \mu$  and  $f(E_0|C_0) \subseteq E$ .*

*Proof.* This follows from the proof of Dye's Theorem, see, e.g., [KM04, Section 7] and [Kec94, 5.26].  $\square$

If  $E$  is compressible, then the result follows from [Proposition 2.1.2](#), (ii). Otherwise by Nadkarni's Theorem (see, e.g., [\[Kec21b, 4.C\]](#))  $\text{EINV}_E$  is nonempty. Consider then the ergodic decomposition  $\{X_e\}_{e \in \text{EINV}_E}$  of  $E$  (see, e.g., [\[Kec21b, 4.11\]](#)). For each  $e \in \text{EINV}_E$ , by [Lemma 2.2.4](#), there is an  $E$ -invariant Borel set  $X_{0,e} \subseteq X_e$  with  $e(X_{0,e}) = 1$ , an  $E_0$ -invariant Borel set  $C_{0,e} \subseteq 2^{\mathbb{N}}$  with  $\mu_0(C_{0,e}) = 1$  and a Borel isomorphism  $f_e: C_{0,e} \rightarrow X_{0,e}$  such that  $(f_e)_*\mu_0 = e$  and  $F_e = f_e(E_0|C_{0,e}) \subseteq E$ . Note that  $F_e$  admits a unique ergodic, invariant measure, namely  $e$ .

The proof of [Lemma 2.2.4](#) is effective enough (see, e.g., the proof of [\[DJK94, 9.6\]](#)), so that  $X_0 = \bigcup_e X_{0,e}$  is Borel and  $F_0 = \bigcup_e F_e$ , which lives on  $X_0$ , is also Borel and hyperfinite. Let  $X' = X \setminus X_0$ . Then by the properties of the ergodic decomposition  $F|X'$  is compressible, so by the compressible case above there is a hyperfinite compressible equivalence relation  $F' \subseteq E|X'$ . Finally put  $F = F_0 \cup F'$ . This clearly works.  $\square$

Recall that the classification theorem for hyperfinite CBER, see [\[DJK94, 9.1\]](#), shows that, up to Borel isomorphism,  $\mathcal{AH}$  consists exactly of the following equivalence relations, where  $E_t$  is the equivalence relation on  $2^{\mathbb{N}}$  given by  $x E_t y \iff \exists m \exists n \forall k (x_{m+k} = y_{n+k})$ :

$$\mathbb{R}I_{\mathbb{N}}, E_t, E_0, 2E_0, 3E_0, \dots, \mathbb{N}E_0, \mathbb{R}E_0.$$

Moreover  $|\text{EINV}_E|$ , for  $E$  in this list, is respectively  $0, 0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}$ .

Below for a quasi-order  $\preceq$  with strict part  $\prec$  on a set  $Q$  and  $q, r \in Q$ , we say that  $r$  is a successor to  $q$  if  $q \prec r$  and  $(s \prec r \implies s \preceq q)$ .

We now have:

**Theorem 2.2.5.** (i)  $\mathbb{R}E_0 \subset_B \mathbb{N}E_0 \subset_B \dots \subset_B 3E_0 \subset_B 2E_0 \subset_B E_0 \subset_B E_t$ , each equivalence relation in this list is a successor in  $\subset_B$  of the one preceding it and  $\mathbb{N}E_0$  is the infimum in  $\subset_B$  of the  $nE_0, n \in \mathbb{N} \setminus \{0\}$ .

(ii)  $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$  and  $E_t$  is a successor of  $\mathbb{R}I_{\mathbb{N}}$  in  $\subset_B$ .

*Proof.* (i) Clearly  $E_0 \subseteq E_t$  and thus  $E_0 \subset_B E_t$  as  $E_0$  is not compressible. To see that  $2E_0 \subset_B E_0$ , note that  $2^{\mathbb{N}} = X_0 \sqcup X_1$ , where  $X_i = \{x \in 2^{\mathbb{N}} : x_0 = i\}$ , and  $E_0|X_i \cong_B E_0$ . From this it follows immediately that  $(n+1)E_0 \subset_B nE_0$ , for each  $n \in \mathbb{N}, n \geq 1$ .

To show that  $\mathbb{N}E_0 \subset_B nE_0$ , for each  $n \in \mathbb{N} \setminus \{0\}$ , it is enough to show that  $\mathbb{N}E_0 \subset_B E_0$ . Let  $s_n = 1^n 0$  be the finite sequence starting with  $n$  1's followed by one 0, for  $n \in \mathbb{N}$ . Let  $X_n$  be the subset of  $2^{\mathbb{N}}$  consisting of all sequences starting with  $s_n$ , let  $\bar{1}$  be the constant 1 sequence and put  $X = 2^{\mathbb{N}} \setminus \{\bar{1}\}$ . Then  $X = \bigsqcup_n X_n$  and  $E_0 \cong_B E|X \cong_B E|X_n$ , for each  $n \in \mathbb{N}$ , which completes the proof that  $\mathbb{N}E_0 \subset_B E_0$ .



Finally to show that  $\mathbb{R}E_0 \subseteq_B \mathbb{N}E_0$ , it is enough to show that  $\mathbb{R}E_0 \subseteq_B E_0$ . To prove this, let for each  $y \in 2^{\mathbb{N}}$ ,  $X_y = \{x \in 2^{\mathbb{N}} : \forall n \in \mathbb{N}(x_{2n} = y_n)\}$ . Then  $2^{\mathbb{N}} = \bigsqcup_y X_y$  and  $E_0|X_y \cong_B E_0, \forall y \in 2^{\mathbb{N}}$ , which immediately implies that  $\mathbb{R}E_0 \subseteq_B E_0$ .

This establishes the non-strict orders in the list of (i). The strict orders and the last two statements of (i) now follow from [Proposition 2.1.3](#).

(ii) Since  $E_t$  is compressible and not smooth, by [Proposition 2.1.2](#),  $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$ . It is also clear that  $E_t$  is a successor of  $\mathbb{R}I_{\mathbb{N}}$ .  $\square$

The following is an immediate corollary of [Theorem 2.2.5](#):

**Corollary 2.2.6.** *Let  $E, F \in \mathcal{A}\mathcal{H}$ . Then*

$$E \subseteq_B F \iff |\text{EINV}_E| \geq |\text{EINV}_F|.$$

The next result is a version of the Glimm-Effros Dichotomy, see [[Kec21b](#), 5.5], for the inclusion order  $\subseteq_B$  instead of  $\sqsubseteq_B$ . It is an immediate corollary of [Theorem 2.2.5](#) and [Theorem 2.2.3](#).

**Corollary 2.2.7.** *Let  $E \in \mathcal{A}\mathcal{E}$ . Then exactly one of the following holds:*

- (i)  $E$  is smooth,
- (ii)  $\mathbb{R}E_0 \subseteq_B E$ .

Denote by  $E_{\infty}$  a universal CBER, in the sense that every CBER  $F$  satisfies  $F \sqsubseteq_B E_{\infty}$ , see, e.g., [[Kec21b](#), 5.C]. Then, by [Corollary 2.2.7](#),  $\mathbb{R}E_0$  is a  $\subseteq_B$ -minimum among all the non-smooth relations in  $\mathcal{A}\mathcal{E}$  and, by [Theorem 2.1.4](#),  $E_{\infty} \times I_{\mathbb{N}}$  is a  $\subseteq_B$ -maximum relation in  $\mathcal{A}\mathcal{E}$ .

## 2.3 A global decomposition

For each cardinal  $\kappa \in \{0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$ , let  $\mathcal{A}\mathcal{E}_{\kappa}$  be the class of all  $E \in \mathcal{A}\mathcal{E}$  such that  $|\text{EINV}_E| = \kappa$ . Clearly  $\mathcal{A}\mathcal{E} = \bigsqcup_{\kappa} \mathcal{A}\mathcal{E}_{\kappa}$  and each  $\mathcal{A}\mathcal{E}_{\kappa}$  is invariant under the equivalence relation associated with the quasi-order  $\subseteq_B$ , by [Proposition 2.1.3](#). We also let for  $\kappa > 0$ ,  $\kappa E = SE$ , where  $S$  is a standard Borel space of cardinality  $\kappa$ .

**Proposition 2.3.1.** (i)  $\mathbb{R}I_{\mathbb{N}}$  is  $\subseteq_B$ -minimum in  $\mathcal{A}\mathcal{E}_0$  and  $E_t$  is  $\subseteq_B$ -minimum among the non-smooth elements of  $\mathcal{A}\mathcal{E}_0$ . (B. Miller) Also  $E_{\infty} \times I_{\mathbb{N}}$  is  $\subseteq_B$ -maximum in  $\mathcal{A}\mathcal{E}_0$ .

(ii) For each  $\kappa > 0$ ,  $\kappa E_0$  is a  $\subseteq_B$ -minimum element of  $\mathcal{A}\mathcal{E}_{\kappa}$  but  $\mathcal{A}\mathcal{E}_{\kappa}$  has no  $\subseteq_B$ -maximum element.

(iii) Let  $\kappa \leq \lambda$ . Then for every  $E \in \mathcal{A}\mathcal{E}_{\lambda}$ , there is  $F \in \mathcal{A}\mathcal{E}_{\kappa}$  such that  $E \subseteq_B F$ .

(iv) (with R. Chen) The map  $E \mapsto E \oplus E_0$  is an order embedding of the non-smooth elements of  $\mathcal{A}\mathcal{E}$  into  $\mathcal{A}\mathcal{E}$ , i.e., for non-smooth  $E, F \in \mathcal{A}\mathcal{E}$ ,  $E \subseteq_B F \iff$



$E \oplus E_0 \subseteq_B F \oplus E_0$ . It maps  $\mathcal{AE}_\kappa$  into  $\mathcal{AE}_{\kappa+1}$ , if  $\kappa$  is finite, and  $\mathcal{AE}_\kappa$  into itself, if  $\kappa$  is infinite.

*Proof.* (i) That  $\mathbb{R}I_{\mathbb{N}}$  is  $\subseteq_B$ -minimum in  $\mathcal{AE}_0$  follows from [Proposition 2.1.2](#) and that  $E_\infty \times I_{\mathbb{N}}$  is  $\subseteq_B$ -maximum in  $\mathcal{AE}_0$  follows from [Theorem 2.1.4](#). Finally we have to show that if  $E \in \mathcal{AE}_0$  is not smooth, then  $E_t \subseteq_B E$ .

Since  $E$  is not smooth, we have that  $E_t \sqsubseteq_B E$  (see [[Kec21b](#), 5.5 and 7.3]), so, as  $E_t$  is compressible,  $E_t \sqsubseteq_B^i E$  (see [[Kec21b](#), 2.27]), i.e.,  $E_t$  is Borel isomorphic to the restriction of  $E$  to an  $E$ -invariant Borel set. So if  $E$  lives on  $X$ , we have a Borel partition  $X = Y \sqcup Z$  into  $E$ -invariant Borel sets such that  $E|Y \cong_B E_t$ . Since  $E|Z$  is compressible, we see, using [Lemma 2.1.5](#), that  $E_t \subseteq_B E_t \oplus E|Z \cong_B E|Y \oplus E|Z \cong_B E$ .

(ii) The fact that  $\kappa E_0$  is a  $\subseteq_B$ -minimum element of  $\mathcal{AE}_\kappa$  is clear from [Theorem 2.2.3](#). That  $\mathcal{AE}_\kappa$  has no  $\subseteq_B$ -maximum element can be seen as follows.

Assume that  $E$  is such a  $\subseteq_B$ -maximum, towards a contradiction. Say  $E$  lives on the space  $X$ . Fix an invariant measure  $\mu$  for  $E$ . We will show that every infinite countable group  $\Gamma$  embeds algebraically into  $[E]$ , the measure theoretic full group of  $E$  with respect to  $\mu$ , contradicting a result of Ozawa, see [[Kec10](#), page 29].

The group  $\Gamma$  admits a free Borel action on a standard Borel space  $Y$ , with associated equivalence relation  $G$  that has exactly  $\kappa$  ergodic, invariant measures. To see this, consider the free part of the shift action of  $\Gamma$  on  $2^\Gamma$ , which has  $2^{\aleph_0}$  ergodic components, and restrict the action to  $\kappa$  many ergodic components. Since  $E$  is  $\subseteq_B$ -maximum in  $\mathcal{AE}_\kappa$ , let  $f: Y \rightarrow X$  be a Borel isomorphism such that  $f(G) = F \subseteq E$ . Then  $\Gamma$  acts freely in a Borel way on  $X$  inducing  $F$ , so that  $\Gamma$  can be algebraically embedded in  $[F]$ , the measure theoretic full group of  $F$  with respect to  $\mu$  (which is clearly invariant for  $F$ ). But  $[F] \leq [E]$ , so  $\Gamma$  embeds algebraically into  $[E]$ .

(iii) We can of course assume that  $\kappa > 0$ . Let  $E \in \mathcal{AE}_\lambda$ . Let  $\{X_e\}_{e \in \text{EINV}_E}$  be the ergodic decomposition of  $E$ , which has  $\lambda$  many components. If  $E$  lives on  $X$ , let  $Y$  be a Borel  $E$ -invariant subset of  $X$  consisting of exactly  $\kappa$  many ergodic components. Put  $Z = X \setminus Y$ . Then let  $E' = E|Y$  and let  $G$  be a compressible equivalence relation on  $Z$  with  $G \supseteq E|Z$ . Let  $F = E' \cup G$ . Then  $E \subseteq F$  and  $F \in \mathcal{AE}_\kappa$ .

(iv) We show that  $E \mapsto E \oplus E_0$  is an order embedding on non-smooth aperiodic CBERs (on uncountable standard Borel spaces). (Note that the only failure is that  $E_t \oplus E_0 \cong_B \mathbb{R}I_{\mathbb{N}} \oplus E_0$ .)

Clearly, if  $E \subseteq_B F$ , then  $E \oplus E_0 \subseteq_B F \oplus E_0$ . Conversely, suppose that  $E \oplus E_0 \subseteq_B F \oplus E_0$ . We want to show that  $E \subseteq_B F$ .

We can write  $E \cong_B R \oplus R'$  and  $E_0 \cong_B S \oplus S'$  with  $R \oplus S \subseteq_B F$  and  $R' \oplus S' \subseteq_B E_0$ . Note that  $R', S, S'$  are all aperiodic hyperfinite (maybe on a countable space), and

since  $E_0 \cong_B S \oplus S'$ , exactly one of  $S$  or  $S'$  must be  $E_0$ , and the other is compressible hyperfinite. Also since  $E$  is non-smooth, we have  $E \cong_B E \oplus E_t$ , and similarly for  $F$ .

We have two cases:

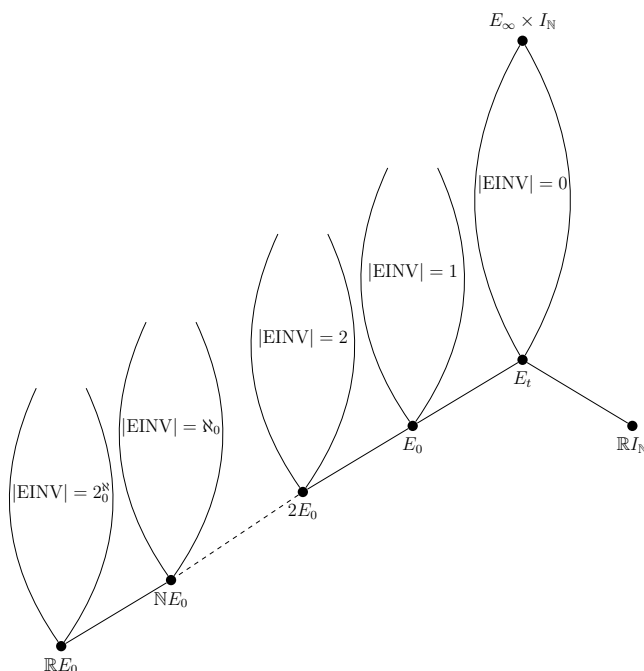
1. If  $S = E_0$  and  $S'$  is compressible, then since  $R' \oplus S' \subseteq_B E_0$ , we must have  $R' \subseteq_B E_0 = S$ .
2. If  $S$  is compressible, then we have  $R' \oplus E_t \subseteq_B S \oplus E_t$ , since  $R' \oplus E_t$  is hyperfinite and  $S \oplus E_t \cong_B E_t$ .

In both cases, we get:

$$E \cong_B E \oplus E_t \cong_B R \oplus R' \oplus E_t \subseteq_B R \oplus S \oplus E_t \subseteq_B F \oplus E_t \cong_B F$$

□

The following picture illustrates parts (i) and (ii) of [Proposition 2.3.1](#).



It is interesting to consider the problem of existence of  $\subseteq_B$ -maximum elements in  $\mathcal{E}_\kappa = \mathcal{A}\mathcal{E}_\kappa \cap \mathcal{E}$  for other classes  $\mathcal{E} \subseteq \mathcal{A}\mathcal{E}$ . This is clearly the case if  $\kappa = 0$  and  $\mathcal{E}$  satisfies the conditions of [Theorem 2.1.4](#), so we will consider  $\kappa \geq 1$ .

Clearly  $\kappa E_0$  is  $\subseteq_B$ -maximum in  $\mathcal{A}\mathcal{H}_\kappa$ . Denote by  $\mathcal{AT}$  the subclass of  $\mathcal{A}\mathcal{E}$  consisting of the treeable equivalence relations.

**Problem 2.3.2.** *Let  $\kappa \geq 1$ . Does  $\mathcal{AT}_\kappa$  have a  $\subseteq_B$ -maximum element?*

If  $E$  is  $\subseteq_B$ -maximum in  $\mathcal{AT}_1$ , then  $\kappa E$  is  $\subseteq_B$ -maximum in  $\mathcal{AT}_\kappa$ , for every  $1 \leq \kappa \leq \aleph_0$ , so we will concentrate in the case  $\kappa = 1$ , i.e., the class of **uniquely ergodic** elements of  $\mathcal{AT}$ . We do not know the answer to this problem but we would like to point out that a positive answer has an implication in the context of the theory of measure preserving CBER, see [Kec21a].

Fix a standard Borel space  $X$  and a measure  $\mu$  on  $X$ . We will consider as in [Kec21a] **pmp** CBER on  $X$ , i.e.,  $\mu$ -measure preserving CBER on  $X$ , where we identify two such relations if they agree  $\mu$ -a.e. Inclusion of pmp relations is also understood in the  $\mu$ -a.e. sense. Such a relation is treeable if it has this property  $\mu$ -a.e. We also denote by  $\text{Aut}(X, \mu)$  the group of measure preserving automorphisms of  $(X, \mu)$ .

**Proposition 2.3.3.** *If  $E$  on a standard Borel space  $X$  is a  $\subseteq_B$ -maximum uniquely ergodic, equivalence relation in  $\mathcal{AT}$ , with (unique) invariant measure  $\mu$ , then for every treeable pmp relation  $F$  on  $(X, \mu)$ , there is an automorphism  $T \in \text{Aut}(X, \mu)$  such that  $T(F) \subseteq E$ .*

*Proof.* We will use the following lemma.

**Lemma 2.3.4.** *Let  $G$  be a treeable pmp CBER on  $(X, \mu)$ . Then there is an ergodic, treeable pmp CBER  $H$  on  $(X, \mu)$  with  $G \subseteq H$ .*

*Proof.* For each  $T \in \text{Aut}(X, \mu)$  denote by  $E_T$  the equivalence relation induced by  $T$ . By [CM14, Theorem 8] the set of  $T \in \text{Aut}(X, \mu)$  such that  $E_T$  is independent of  $G$  (see [KM04, Section 27] for the notion of independence) is comeager in  $\text{Aut}(X, \mu)$ , equipped with the usual weak topology. So is the set of all ergodic  $T \in \text{Aut}(X, \mu)$ , see [Kec10, Theorem 2.6]. Thus there is an ergodic  $T \in \text{Aut}(X, \mu)$  such that  $E_T$  is independent of  $G$ . Then put  $H = E_T \vee G$ , the smallest equivalence relation containing  $E_T$  and  $G$ .  $\square$

By Lemma 2.3.4, we can assume that  $F$  is ergodic. We can also assume that there is  $F' \in \mathcal{AT}$  which agrees with  $F$   $\mu$ -a.e. By considering the ergodic decomposition of  $F'$ , we can also assume that  $\mu$  is the unique invariant measure for  $F'$ . Fix then a Borel automorphism  $T: X \rightarrow X$  such that  $T(F') \subseteq E$ . Then both  $T_*\mu$  and  $\mu$  are  $T(F')$ -invariant. Since  $T(F')$  is uniquely ergodic, it follows that  $T_*\mu = \mu$ , i.e.,  $T \in \text{Aut}(X, \mu)$  and the proof is complete.  $\square$

**Remark 2.3.5.** We note here that an analog of the conclusion of Proposition 2.3.3 is valid for the class  $\mathcal{AH}$ . More precisely, let  $X = 2^{\mathbb{N}}$  and let  $\mu$  be the usual product

measure on  $X$ . Then for every hyperfinite pmp relation  $F$  on  $(X, \mu)$ , there is an automorphism  $T \in \text{Aut}(X, \mu)$  such that  $T(F) \subseteq E_0$ . This can be seen as follows: By [Kec10, 5.4] (in which the aperiodicity of  $E$  is not needed), we can find a hyperfinite pmp relation  $F'$  such that  $F \subseteq F'$ . By Dye's Theorem (see, e.g., [Kec10, 3.13]) there is an automorphism  $T \in \text{Aut}(X, \mu)$  such that  $T(F') = E_0$  and thus  $T(F) \subseteq E_0$ .

### 3 Topological realizations

#### 3.1 Dense realizations and Lusin marker schemes

We will first use the results in Section 2 to prove the following:

**Theorem 3.1.1.** *For every equivalence relation  $E \in \mathcal{AE}$  and every perfect Polish space  $Y$ , there is a minimal topological realization of  $E$  in  $Y$ .*

*Proof.* First, since for every perfect Polish space  $Y$  there is a continuous bijection from the Baire space  $\mathbb{N}^{\mathbb{N}}$  onto  $Y$  (see [Kec95, 7.15]), we can assume that  $Y = \mathbb{N}^{\mathbb{N}}$ . Moreover by Corollary 2.2.7, it is enough to prove this result for  $E = \mathbb{R}E_0$  and  $E = \mathbb{R}I_{\mathbb{N}}$ .

*Case 1:  $\mathbb{R}E_0$ .*

Consider the shift map of  $\mathbb{Z}$  on  $2^{\mathbb{Z}}$  with associated equivalence relation  $F'$ . Let  $Y = \{x \in 2^{\mathbb{Z}} : [x]_{F'} \text{ is dense in } 2^{\mathbb{Z}}\}$ . Clearly  $Y$  is a dense, co-dense  $G_{\delta}$  set in  $2^{\mathbb{Z}}$ , so, in particular, it is a zero-dimensional Polish space (with the relative topology from  $2^{\mathbb{Z}}$ ). We next check that every compact set in  $Y$  has empty interior. Indeed let  $K \subseteq Y$  be compact in  $Y$ . Then  $K$  is compact in  $2^{\mathbb{Z}}$ . If now  $V$  is open in  $2^{\mathbb{Z}}$  and  $\emptyset \neq V \cap Y \subseteq K$ , then since  $Y$  is dense in  $2^{\mathbb{Z}}$ , by looking at  $V \setminus K$  we see that  $V \subseteq K$ , contradicting that  $Y$  is also co-dense in  $2^{\mathbb{Z}}$ .

By [Kec95, 7.7]  $Y$  is homeomorphic to  $\mathbb{N}^{\mathbb{N}}$ . Moreover if  $F = F'|_Y$ ,  $F$  has dense classes and  $|\text{EINV}_F| = 2^{\aleph_0}$ , so  $F \cong_B \mathbb{R}E_0$ .

*Case 2:  $\mathbb{R}I_{\mathbb{N}}$ .*

Consider the equivalence relation  $R$  on  $\mathbb{N}^{\mathbb{N}}$  given by

$$xRy \iff \exists m \forall n \geq m (x_n = y_n).$$

Let  $A \subseteq \mathbb{N}^{\mathbb{N}}$  be an uncountable Borel partial transversal for  $R$  (i.e., no two distinct elements of  $A$  are in  $R$ ). Then, as  $R$  is not smooth, denoting by  $B = [A]_R$  the  $R$ -saturation of  $A$ , we also have that  $Y = \mathbb{N}^{\mathbb{N}} \setminus B$  is uncountable. Fix then a Borel bijection  $f: A \rightarrow Y$  and let  $F$  be the equivalence relation obtained by adding to each  $[a]_R$ ,  $a \in A$ , the point  $f(a)$ . Then  $F$  is a smooth CBER, so  $F \cong_B \mathbb{R}I_{\mathbb{N}}$ , and every  $F$ -class is dense in  $\mathbb{N}^{\mathbb{N}}$ .  $\square$

A **complete section** of an equivalence relation  $E$  on  $X$  is a subset  $Y \subseteq X$  which meets every  $E$ -class. Recall that a **vanishing sequence of markers** for a CBER  $E$  is a decreasing sequence of complete Borel sections  $\{A_n\}$  for  $E$  such that  $\bigcap_n A_n = \emptyset$ . A very useful result in the theory of CBER is the Marker Lemma, which asserts that every  $E \in \mathcal{AE}$  admits a vanishing sequence of markers, see, e.g., [Kec21b, 2.15]. We will see next that [Theorem 3.1.1](#) implies a strong new version of a marker lemma.

**Definition 3.1.2.** Let  $E$  be a CBER on a standard Borel space  $X$ . A **Lusin marker scheme** for  $E$  is a family  $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$  of Borel sets such that

- (i)  $A_\emptyset = X$ ;
- (ii)  $\{A_{sn}\}_n$  are pairwise disjoint and  $\bigsqcup_n A_{sn} \subseteq A_s$ ;
- (iii) Each  $A_s$  is a complete section for  $E$ .

We have two types of Lusin marker schemes:

- 1) The Lusin marker scheme  $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$  for  $E$  is of **type I** if in (ii) above we actually have that  $\bigsqcup_n A_{sn} = A_s$  and moreover the following holds:

- (iv) For each  $x \in \mathbb{N}^{\mathbb{N}}$ ,  $\bigcap_n A_{x|n}$  is a singleton.

Then in this case, for each  $x \in \mathbb{N}^{\mathbb{N}}$ ,  $A_n^x = A_{x|n} \setminus \bigcap_n A_{x|n}$  is a vanishing sequence of markers.

- 2) The Lusin marker scheme  $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$  for  $E$  is of **type II** if it satisfies the following:

- (v) If for each  $n$ ,  $B_n = \bigsqcup\{A_s : s \in \mathbb{N}^n\}$ , then  $\{B_n\}$  is a vanishing sequence of markers.

**Theorem 3.1.3.** *Every  $E \in \mathcal{AE}$  admits a Lusin marker scheme of type I and a Lusin marker scheme of type II.*

*Proof.* Type I: By [Theorem 3.1.1](#), we can assume that  $E$  lives on  $\mathbb{N}^{\mathbb{N}}$  and that every equivalence class is dense. Let then for each  $s \in \mathbb{N}^n$ ,  $A_s = \{x : x|n = s\}$ .

Type II: By [Theorem 3.1.1](#), we can assume that  $E$  lives on  $\mathbb{R}$  and that every equivalence class is dense. By induction on  $n$ , we can easily construct open sets  $A_s$ ,  $s \in \mathbb{N}^n$ , such that  $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$  is a Lusin marker scheme for  $E$  and moreover it has the following properties:

- (a) Each  $A_s$ ,  $s \in \mathbb{N}^n$ ,  $n \geq 1$ , is contained in  $(n, \infty)$ ;

(b) Each  $A_s$ ,  $s \in \mathbb{N}^n$ ,  $n \geq 1$ , has non-empty intersection with the interval  $(k, k+1)$  for every  $k \geq n$ .

Then clearly  $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$  is of type II. □

**Remark 3.1.4.** (a) We can also easily see that every  $E \in \mathcal{AE}$  admits a **Cantor marker scheme**  $\{A_s\}_{s \in 2^{<\mathbb{N}}}$  of each type, which is defined in an analogous way.

(b) By applying [Theorem 3.1.3](#) to  $\mathbb{R}E$ , and using the ccc property for category, we can see that every  $E \in \mathcal{AE}$  admits a variant of a Lusin marker scheme of type I, where condition (iv) in [Definition 3.1.2](#) is replaced by the following condition:

(iv)' For each  $x \in \mathbb{N}^{\mathbb{N}}$ ,  $\bigcap_n A_{x|n}$  has at most one element and for a comeager set of  $x$  it is empty.

## 3.2 Continuous action realizations

Any CBER has a continuous action realization, i.e., a topological realization induced by a continuous action of a countable group on a Polish space. We will consider what additional properties of the action and the Polish space of the realization are possible. For example, we have the following:

**Proposition 3.2.1.** *Every  $E \in \mathcal{AE}$  has a continuous action realization in the Baire space  $\mathbb{N}^{\mathbb{N}}$ .*

*Proof.* By the usual change of topology arguments, we can assume that  $E$  is induced by a continuous action of a countable group on a 0-dimensional space  $X$ . Let  $P \subseteq X$  be the perfect kernel of  $X$ , which is clearly invariant under the action. Since  $X \setminus P$  is countable, it is easy to see that  $E|P \cong_B E$ , so we can assume that  $X$  is perfect. Let then  $D$  be a countable dense subset of  $X$  which is also invariant under the action and put  $Y = X \setminus D$ . Then again  $E \cong_B E|Y$ . The space  $Y$  is a nonempty, 0-dimensional Polish space in which every compact set has empty interior and thus is homeomorphic to the Baire space (see [[Kec95](#), Theorem 7.7]). □

**Definition 3.2.2.** (i) A **transitive action realization**, resp., **minimal action realization** of a CBER is a topological realization induced by a continuous, topologically transitive action of a countable group (i.e., one which has a dense orbit), resp., induced by a continuous, topologically minimal action of a countable group (i.e., one for which all orbits are dense).

(ii) A  **$\sigma$ -compact action realization**, resp., **locally compact action realization**, resp., **compact action realization** of a CBER is a topological realization induced by a continuous action of a countable group on a  $\sigma$ -compact, resp., locally compact, resp., compact Polish space.

(iii) A **transitive,  $\sigma$ -compact action realization** is a topological realization induced by a continuous, topologically transitive action of a countable group on a  $\sigma$ -compact Polish space. Similarly we define the concepts of

**transitive, locally compact action realization,**  
**transitive, compact action realization,**  
**minimal,  $\sigma$ -compact action realization,**  
**minimal, locally compact action realization,**  
**minimal, compact action realization.**

We first note the following fact:

**Proposition 3.2.3.** *If  $E \in \mathcal{AE}$  has a compact action realization or a transitive action realization on a perfect Polish space or a minimal action realization, then  $E$  is not smooth.*

*Proof.* Suppose a smooth  $E$  has a compact action realization  $F$ , towards a contradiction. Then there is a compact invariant subset  $K$  in which the action is minimal. Since  $F|_K$  is also smooth, by [Kec95, 8.46] some orbit in  $K$  is non-meager in  $K$ , thus consists of isolated points in  $K$ . Minimality then implies that  $K$  consists of a single infinite orbit, contradicting compactness.

The proof of the case of a transitive action realization on a perfect Polish space or a minimal action realization follows also from [Kec95, 8.46].  $\square$

We first note here that the hypothesis of perfectness in [Proposition 3.2.3](#) is necessary.

**Proposition 3.2.4.** *Every smooth equivalence relation in  $\mathcal{AE}$  has a transitive locally compact action realization (in some non-perfect space).*

*Proof.* Let  $\mathbb{N} = \bigsqcup_{q \in \mathbb{Q}} N_q$  be a decomposition of  $\mathbb{N}$  into infinite sets indexed by the rationals. Define then recursively  $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ , with  $\text{Im } z_n > 0$ ,  $\text{Im } z_{n+1} < \text{Im } z_n$ ,  $\text{Im } z_n \rightarrow 0$ , and pairwise disjoint closed squares  $S_n$  with center  $z_n$  with  $\text{Im } S_n > 0$  as follows:

If  $0 \in N_q$ , choose  $z_0 \in \{q\} \times \mathbb{R}$  and let  $S_0$  be a very small square around  $z_0$ . At stage  $n+1$ , if  $n+1 \in N_q$ , choose  $z_{n+1} \in \{q\} \times \mathbb{R}$  so that  $0 < \text{Im } z_n < \frac{1}{n+1}$ ,  $\text{Im } z_{n+1} < \text{Im } z_n$ ,  $z_{n+1} \notin \bigcup_{m \leq n} S_m$ , and then choose  $S_{n+1}$  to be a small square around  $z_{n+1}$  so that it has empty intersection with all  $S_m$ ,  $m \leq n$ .

Put  $X = \mathbb{R} \cup \{z_n\}_{n \in \mathbb{N}}$ . Then  $X$  is closed in  $\mathbb{C}$ , so it is locally compact. Next define  $T: X \rightarrow X$  as follows:

If  $x \in \mathbb{R}$ , then  $T(x) = x + 1$ . If  $x = z_n$  with  $n \in N_q$ , so that  $x \in \{q\} \times \mathbb{R}$ , and if in the increasing enumeration of  $N_q$ ,  $n$  is the  $i$ th element, then put  $T(x) = z_m$ , where

$m$  is the  $i$ th element in the increasing enumeration of  $N_{q+1}$ . It is not hard to check that  $T$  is a homeomorphism of  $X$ . For example, to check that  $T$  is continuous (a similar argument works for  $T^{-1}$ ) let  $w_n, w \in X$ , with  $w_n \rightarrow w$ , in order to show that  $T(w_n) \rightarrow T(w)$ . We can assume of course that  $w_n \notin \mathbb{R}, w \in \mathbb{R}, \text{Im } w_n \rightarrow 0$ . Now  $\text{Re } T(w_n) = \text{Re } w_n + 1$  and  $\text{Im } T(w_n) \rightarrow 0$ , thus  $T(w_n) = \text{Re } w_n + 1 + i \text{Im } T(w_n) \rightarrow w + 1 = T(w)$ .

Next for each pair  $(m, n) \in \mathbb{N}^2$ , let  $T_{m,n}$  be the homeomorphism of  $X$  that switches  $z_m$  with  $z_n$  and keeps every other point of  $X$  fixed. Then the group generated by all  $T_{m,n}$  and  $T$  acts continuously on  $X$ . One of its orbits is  $\{z_n\}$  which is dense in  $X$ , thus the action is topologically transitive. The equivalence relation  $F$  it generates has as classes the set  $\{z_n\}$  and the sets of the form  $x + \mathbb{Z}$ , for  $x \in \mathbb{R}$ , so it is aperiodic and smooth, with transversal  $\{z_0\} \cup [0, 1)$ .  $\square$

Also the hypothesis of compactness in [Proposition 3.2.3](#) is necessary.

**Proposition 3.2.5.** *Every smooth equivalence relation in  $\mathcal{AE}$  has a locally compact action realization on a perfect space, in fact one in the space  $2^{\mathbb{N}} \setminus \{\bar{1}\}$ , where  $\bar{1}$  is the constant 1 sequence.*

*Proof.* We use an example in [[DJK94](#), page 200, (b)]. Consider the space  $X = 2^{\mathbb{N}} \setminus \{\bar{1}\}$ . For each  $m \neq n$ , let  $h_{m,n}$  be the homeomorphism of  $X$  defined by:  $h_{m,n}(1^m 0^{\wedge} y) = 1^n 0^{\wedge} y, h_{m,n}(1^n 0^{\wedge} y) = 1^m 0^{\wedge} y, h_{m,n}(x) = x$ , otherwise. Then the group generated by these homeomorphisms acts continuously on  $X$  and generates the equivalence relation  $F$  given by:  $x F y \iff \exists z (x = 1^m 0^{\wedge} z \ \& \ y = 1^n 0^{\wedge} z)$ , which is smooth aperiodic.  $\square$

We next show that non-smooth hyperfinite equivalence relations in  $\mathcal{AE}$  have the strongest kind of topological realization. For a countable group  $\Gamma$ , recall that a subshift of  $2^{\Gamma}$  is the restriction of the shift action of  $\Gamma$  to a nonempty closed invariant subset.

**Theorem 3.2.6.** *Every non-smooth hyperfinite equivalence relation in  $\mathcal{AE}$  has a minimal, compact action realization on the Cantor space  $2^{\mathbb{N}}$ . In fact, we have the following:*

- (i) *If it is compressible, then it can be realized by a minimal subshift of  $2^{\mathbb{F}_2}$ .*
- (ii) *If it is not compressible, then it can be realized by a minimal subshift of  $2^{\mathbb{Z}}$ .*

*Proof.* (i) Consider  $E_t$ . Then  $E_t$  is generated by a continuous action of  $\mathbb{F}_2$ , see [[Kec21b](#), 2.B], defined as follows: The first generator acts via  $i^{\wedge} x \mapsto (1 - i)^{\wedge} x$ , and



the second generator acts via

$$\begin{aligned} 0 \hat{x} &\mapsto 00 \hat{x} \\ 11 \hat{x} &\mapsto 1 \hat{x} \\ 10 \hat{x} &\mapsto 01 \hat{x} \end{aligned}$$

This action has a clopen 2-generator, namely the partition given by  $\{X_0 = 0 \hat{2}^{\mathbb{N}}, X_1 = 1 \hat{2}^{\mathbb{N}}\}$ . This means that the sets  $\gamma \cdot X_i, \gamma \in \mathbb{F}_2, i \leq 1$ , separate points. This implies that this action is (topologically) isomorphic to a subshift of  $2^{\mathbb{F}_2}$ .

(ii) Assume that  $E \in \mathcal{A}\mathcal{H}$  is non-compressible and let  $\kappa = |\text{EINV}_E| > 0$ . By a theorem of Downarowicz [Dow91, Theorem 5], for every metrizable Choquet simplex  $K$  there is a minimal subshift of  $2^{\mathbb{Z}}$  such that  $K$  is affinely homeomorphic to the simplex of invariant measures for this subshift. In particular the cardinality of the set of ergodic, invariant measures for this subshift is the same as the cardinality of the set of extreme points of  $K$ . Fix now a compact Polish space  $X$  of cardinality  $\kappa$  and let  $K$  be the Choquet simplex of measures on  $X$ . The extreme points are the Dirac measures, so there are exactly  $\kappa$  many of them. Thus we can find a minimal subshift of  $2^{\mathbb{Z}}$  with exactly  $\kappa$  many ergodic, invariant measures and therefore if  $F$  is the equivalence relation induced by this subshift, we have that  $E \cong_B F$ .  $\square$

Although  $E_t$  does not have a minimal, compact action realization where the acting group is amenable (otherwise it would have an invariant measure), we have the following:

**Proposition 3.2.7.** *A compressible, non-smooth, hyperfinite CBER has a minimal, locally compact action realization where the acting group is  $\mathbb{Z}$ .*

*Proof.* It is known that there are minimal homeomorphisms on uncountable locally compact spaces with no invariant measure, which thus generate a compressible non-smooth hyperfinite CBER; see, e.g., [Dan01, Section 2]. Below we give a simple example:

Let  $A = \mathbb{Z}/4\mathbb{Z}$  as an abelian group, and let  $X \subseteq A^{\mathbb{N}}$  be the set of sequences which eventually lie in  $\{1, 2\}$ . Let  $X_n = A^n \times \{1, 2\}^{\mathbb{N}}$ , so that  $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$  and  $X = \bigcup_n X_n$ . We give  $X_n$  the usual product topology, so that  $X_n$  is clopen in  $X_{n+1}$ , and  $X$  the inductive limit topology, so that  $U \subseteq X$  is open iff  $\forall n (U \cap X_n$  is open in  $X_n)$ . This is Hausdorff, locally compact and second countable, with basis  $\bigcup_n \mathcal{B}_n$ , where  $\mathcal{B}_n$  is a countable basis for  $X_n$ . Thus  $X$  is a locally compact Polish space, see, e.g., [Kec95, 5.3].

Let now  $\varphi : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$  be the odometer map, i.e., addition by 1 with carry, which is a homeomorphism of  $A^{\mathbb{N}}$ . Note that  $\varphi(X) \subseteq X$  and  $\varphi^{-1}(X) \subseteq X$ . We next check

that  $\varphi|X$  is a homeomorphism of  $X$ . It enough to check that  $\varphi|X_n: X_n \rightarrow X$  and  $\varphi^{-1}|X_n: X_n \rightarrow X$  are continuous. This follows from noticing that  $\varphi(X_n) \subseteq X_{n+1}$  and  $\varphi^{-1}(X_n) \subseteq X_{n+1}$ .

Let  $E$  be the equivalence relation on  $X$  induced by  $\varphi|X$ . Denote by  $E'_0$  the equivalence relation on  $A^{\mathbb{N}}$  defined by  $x E'_0 y \iff \exists m \forall n \geq m (x_n = y_n)$ . Then  $E = E'_0|X$  and  $E|X_n = E'_0|X_n$ , so  $\varphi|X$  is minimal.

Finally, we show that  $E$  is compressible. For every  $x \in X$ , let  $n_x$  be least such that  $x \in X_{n_x}$ , and define the Borel map  $f: X \rightarrow X$  as follows:

$$f(x)_n = \begin{cases} x_n + 2 & n = n_x \\ x_n & n \neq n_x \end{cases}$$

Then  $f$  is a compression of  $E$ . □

**Remark 3.2.8.** Here are also some other minimal, locally compact action realizations of a compressible, non-smooth, hyperfinite CBER (but where the acting group is not  $\mathbb{Z}$ ).

(i) Let  $X$  be the locally compact space constructed in the proof of [Proposition 3.2.4](#), whose notation we use below. For each  $q \in \mathbb{Q}$ , let  $T_q: X \rightarrow X$  be the homeomorphism which is translation by  $q$  on  $\mathbb{R}$  and defined on  $\{z_n\}$  in a way similar to translation by 1 in the proof of [Proposition 3.2.4](#). Also define a homeomorphism  $T: X \rightarrow X$  as follows:  $T$  is the identity on  $\mathbb{R}$ . Next let for each  $q \in \mathbb{Q}$ ,  $N_q = \{n_0^q < n_1^q < n_2^q < \dots\}$  be the increasing enumeration of  $N_q$  and define  $T(z_{n_{2n+3}^q}) = z_{n_{2n+1}^q}$ ,  $T(z_{n_1^q}) = z_{n_0^q}$ ,  $T(z_{n_{2n}^q}) = z_{n_{2n+2}^q}$ ,  $n \in \mathbb{N}$ .

The group generated by  $T, T_q, q \in \mathbb{Q}$  is abelian and acts continuously on  $X$ . The orbits consist of  $\{z_n\}$  and the sets of the form  $x + \mathbb{Q}$  for  $x \in \mathbb{R}$ , so the action is minimal. Finally there is clearly no invariant measure for this action.

(ii) Another construction, where the acting group is actually  $\mathbb{Z}^2$  is the following: Let  $S$  be a minimal homeomorphism on an uncountable compact metric space  $K$ , inducing the equivalence relation  $F$ , and let  $X = K \times \mathbb{Z}$ . Then let  $\mathbb{Z}^2$  act by homeomorphisms on  $X$ , where one of the generators acts like  $S$  on  $K$  and the other as translation by 1 on  $\mathbb{Z}$ . The associated equivalence relation of this action is Borel isomorphic to  $F \times I_N$  so it is compressible, non-smooth and hyperfinite by [[Kec21b](#), 7.27].

Below for a Borel action of a countable group  $\Gamma$  on a standard Borel space  $X$  and a probability measure  $\zeta$  on  $\Gamma$ , we say that a measure  $\mu$  on  $X$  is  **$\zeta$ -stationary** if  $\mu = \int \gamma_* \mu \, d\zeta(\gamma)$ .

It is easy to see that  $\mu$  is quasi-invariant under the action, i.e., the action sends  $\mu$ -null sets to  $\mu$ -null sets. Next we check that if the action has infinite orbits, then  $\mu$

is non-atomic. Let  $x \in X$  be such that  $\mu(\{x\}) > 0$ , towards a contradiction. Since  $\mu(\{x\}) = \int \mu(\gamma^{-1} \cdot \{x\}) d\zeta(\gamma)$ , if  $\mu(\gamma^{-1} \cdot \{x\}) \leq \mu(\{x\}), \forall \gamma$ , then as  $\mu(\gamma^{-1} \cdot \{x\}) > 0$ , we must have that  $\mu(\gamma^{-1} \cdot \{x\}) = \mu(\{x\}), \forall \gamma$ , a contradiction. Thus we see that for every  $x \in X$  with  $\mu(\{x\}) > 0$ , there is  $x' \in \Gamma \cdot x$ , with  $\mu(\{x'\}) > \mu(\{x\})$ . So we can find  $x_0, x_1, x_2, \dots$  with  $\mu(\{x_0\}) < \mu(\{x_1\}) < \mu(\{x_2\}) < \dots$ , a contradiction.

We use these facts and [Theorem 3.2.6](#) to prove the following:

**Proposition 3.2.9.** *Let  $E \in \mathcal{AE}$  be an equivalence relation on a standard Borel space  $X$ . Then the following are equivalent:*

- (i)  *$E$  is not smooth;*
- (ii) *There is a Borel action of a countable group  $\Gamma$  on  $X$  generating  $E$ , such that for every measure  $\zeta$  on  $\Gamma$  there is a  $\zeta$ -stationary, ergodic for this action measure on  $X$ .*
- (iii) *There is a Borel action of a countable group  $\Gamma$  on  $X$  generating  $E$ , such that for some measure  $\zeta$  on  $\Gamma$  there is a  $\zeta$ -stationary, ergodic for this action measure on  $X$ .*

*Proof.* If (iii) holds, then  $E$  admits a non-atomic, ergodic, quasi-invariant measure, so it is not smooth. We next prove that (i) implies (ii).

Since  $E$  is not smooth, by the Glimm-Effros dichotomy, there is an  $E$ -invariant Borel set  $Y \subseteq X$  such that  $E|_Y$  is non-smooth, hyperfinite. Then, by [Theorem 3.2.6](#), there is a continuous action of  $\Gamma = \mathbb{F}_\infty$  on a compact space  $Z$  inducing an equivalence relation  $F \cong_B E|_Y$ . Let  $\zeta$  by any measure on  $\Gamma$ . Then there is a  $\zeta$ -stationary for this action measure on  $Z$ , see, e.g., [\[CKM13\]](#). The set of  $\zeta$ -stationary for this action measures is thus a non-empty compact, convex set of measures, so it has an extreme point which is therefore ergodic. Transferring this back to  $Y$  and extending the  $\Gamma$  action to  $X$  so that it generates  $E|(X \setminus Y)$  on  $X \setminus Y$ , we see that (ii) holds.  $\square$

The following question is open:

**Problem 3.2.10.** *Does every non-smooth  $E \in \mathcal{AE}$  have any of the topological realizations stated in [Definition 3.2.2](#)? In particular, does every non-smooth  $E \in \mathcal{AE}$  admit a compact action realization?*

We will consider the case of compact action realizations in the next two sections. The answer to the following is also unknown:

**Problem 3.2.11.** *If a CBER admits a compact action realization, does it admit one in which the underlying space is  $2^{\mathbb{N}}$ ?*

Recall that a **reduction** of an equivalence relation  $E$  on  $X$  to an equivalence relation  $F$  on  $Y$  is a map  $f: X \rightarrow Y$  such that  $xEy \iff f(x)Ff(y)$ . If such a Borel reduction exists, we say that  $E$  is **Borel reducible** to  $F$  and write  $E \leq_B F$ . If  $E \leq_B F$  and  $F \leq_B E$ , then  $E, F$  are **Borel bireducible**, in symbols  $E \sim_B F$ . We note here that the following weaker version of [Problem 3.2.10](#) is also open:

**Problem 3.2.12.** *Is every non-smooth  $E \in \mathcal{AE}$  Borel bireducible to some  $F \in \mathcal{AE}$  which has any of the topological realizations stated in [Definition 3.2.2](#)? In particular, can one find such an  $F$  that admits a compact action realization?*

### 3.3 Compact action realizations

(A) We have seen in [Theorem 3.2.6](#) that the answer to [Problem 3.2.10](#) is affirmative in the strongest sense for hyperfinite  $E$  but the situation for general  $E$  is unclear. The following results provide some cases of non-hyperfinite equivalence relations that admit compact action realizations.

**Theorem 3.3.1.** *For every infinite countable group  $\Gamma$ ,  $F(\Gamma, 2^{\mathbb{N}})$  admits a compact action realization. If  $\Gamma$  is also finitely generated, then  $E^{ap}(\Gamma, 2^{\mathbb{N}})$  admits a compact action realization. In fact in both cases such a realization can be taken to be a subshift of  $(2^{\mathbb{N}})^{\Gamma}$ .*

*Proof.* The result will follow easily from the following lemma, which is an extended version of the result in [\[Ele18\]](#), who dealt with the case of free actions. Denote below by  $\mathbf{s} = \mathbf{s}_{\Gamma, 3^{\mathbb{N}}}$  the shift action of  $\Gamma$  on  $(3^{\mathbb{N}})^{\Gamma}$  and for each  $y \in (3^{\mathbb{N}})^{\Gamma}$ , let  $\text{stab}(y) \leq \Gamma$  be the stabilizer of  $y$  in this action.

**Lemma 3.3.2.** *Let  $\mathbf{a}$  be a Borel action of an infinite countable group  $\Gamma$  on a standard Borel space  $X$ . Then there is an equivariant Borel embedding  $p: X \rightarrow (3^{\mathbb{N}})^{\Gamma}$  of the action  $\mathbf{a}$  to the shift action  $\mathbf{s}$  of  $\Gamma$  on  $(3^{\mathbb{N}})^{\Gamma}$  such that if  $y \in \overline{p(X)}$ , then there are  $y_0, y_1, \dots \in p(X)$  with*

$$\text{stab}(y) \subseteq \bigcup_m \bigcap_{n \geq m} \text{stab}(y_n).$$

*Proof.* Let  $\Gamma = \{\gamma_n\}$  and let  $\varphi_n: X \rightarrow \{0, 1, 2\}$  be a Borel coloring of the graph of  $\gamma_n^{\mathbf{a}}$ , where  $\gamma_n^{\mathbf{a}}(x) = \mathbf{a}(\gamma_n, x)$  (see [\[KST99, 4.6\]](#)). By changing the topology, we can assume that  $X$  is 0-dimensional, so a  $G_{\delta}$  subset of  $3^{\mathbb{N}}$ , and  $\mathbf{a}, \varphi_n$  are continuous.

Let  $\theta: X \rightarrow 3^{\mathbb{N}}$  be defined by

$$\theta(x) = (x(0), \varphi_0(x), x(1), \varphi_1(x), \dots).$$

Then  $\theta$  is a homeomorphism of  $X$  with a  $G_\delta$  subset of  $3^\mathbb{N}$ , and we have that if  $\gamma_n^{\mathbf{a}} \cdot x = y \neq x$ , then  $\theta(y)(2n+1) \neq \theta(x)(2n+1)$ , where we write  $\gamma_n^{\mathbf{a}} \cdot x$  for  $\gamma_n^{\mathbf{a}}(x)$ .

Thus identifying  $x$  with  $\theta(x)$ , we can thus assume that

- (i)  $X$  is a  $G_\delta$  subset of  $3^\mathbb{N}$ ,
- (ii) The action  $\mathbf{a}$  is continuous,
- (iii)  $\gamma_n^{\mathbf{a}} \cdot x = y \neq x \implies y(2n+1) \neq x(2n+1)$ .

Consider now the standard equivariant Borel embedding of the action  $\mathbf{a}$  into the shift action on  $(3^\mathbb{N})^\Gamma$  given by

$$p: X \rightarrow (3^\mathbb{N})^\Gamma$$

$$x \mapsto (\gamma \mapsto (\gamma^{-1})^{\mathbf{a}} \cdot x).$$

Let now  $y \in \overline{p(X)}$  and take  $x_n \in X$  with  $p(x_n) \rightarrow y$ .

We claim that if  $\gamma \in \Gamma$  is such that for every  $n$ ,  $\gamma^{\mathbf{s}} \cdot p(x_n) \neq p(x_n)$  (equivalently  $\gamma^{\mathbf{a}} \cdot x_n \neq x_n$ ), then  $\gamma^{\mathbf{s}} \cdot y \neq y$ . In fact we will show that  $(\gamma^{\mathbf{s}} \cdot y)(1) \neq y(1)$ , by showing that if  $\gamma = \gamma_m$ , then  $(\gamma^{\mathbf{s}} \cdot y)(1)(2m+1) \neq y(1)(2m+1)$ .

We have  $p(x_n) \rightarrow y$ , so  $p(x_n)(1) \rightarrow y(1)$ , i.e.,  $x_n \rightarrow y(1)$ . Also  $\gamma^{\mathbf{s}} \cdot p(x_n) \rightarrow \gamma^{\mathbf{s}} \cdot y$ , so  $(\gamma^{\mathbf{s}} \cdot p(x_n))(1) \rightarrow (\gamma^{\mathbf{s}} \cdot y)(1)$ , i.e.,  $p(x_n)(\gamma^{-1}) \rightarrow y(\gamma^{-1})$  or  $\gamma^{\mathbf{a}} \cdot x_n \rightarrow (\gamma^{\mathbf{s}} \cdot y)(1)$ . In the discrete space  $\{0, 1, 2\}$  we have  $(\gamma^{\mathbf{a}} \cdot x_n)(2m+1) \rightarrow (\gamma^{\mathbf{s}} \cdot y)(1)(2m+1)$  and  $x_n(2m+1) \rightarrow y(1)(2m+1)$ . Since  $\gamma^{\mathbf{a}} \cdot x_n \neq x_n$ , we have that  $\gamma^{\mathbf{a}} \cdot x_n(2m+1) \neq x_n(2m+1)$  and this concludes the proof of the claim.

Thus we have shown that if  $z_n = p(x_n) \rightarrow y$  and  $\delta \in \text{stab}(y)$ , then for some  $n$ ,  $\delta \in \text{stab}(z_n)$ , and so there are infinitely many  $n$  with this property. Let now  $\text{stab}(y) = \{\delta_m\}$  and by recursion on  $m = 0, 1, 2, \dots$ , define a sequence  $n_0^m < n_1^m < n_2^m < \dots$  such that  $(n_i^{m+1})$  is a subsequence of  $(n_i^m)$  and  $\delta_m \in \text{stab}(z_{n_i^m})$ , for each  $i$ . Let then  $k_m = n_0^m$  be the diagonal subsequence and put  $y_m = z_{k_m}$ . Then for each  $m$ ,  $\delta_m \in \bigcap_{n \geq m} \text{stab}(y_n)$ , so  $\text{stab}(y) \subseteq \bigcup_m \bigcap_{n \geq m} \text{stab}(y_n)$ .  $\square$

From [Lemma 3.3.2](#) it is clear that if  $\mathbf{a}$  is a free action, then  $\overline{p(X)}$  is contained in  $Fr((3^\mathbb{N})^\Gamma)$  and thus in particular by taking  $\mathbf{a}$  to be the restriction of the shift action to  $Fr((3^\mathbb{N})^\Gamma)$ , we see that this action is Borel isomorphic to a subshift contained in  $Fr((3^\mathbb{N})^\Gamma)$ . Of course we can trivially replace  $3^\mathbb{N}$  by  $2^\mathbb{N}$  here, so this proves the first statement of the theorem.

Assume now that  $\Gamma$  is finitely generated. In a similar way, to prove the second statement of the theorem, it is enough to show, in the notation of [Lemma 3.3.2](#), that if the action  $\mathbf{a}$  is aperiodic, so that the stabilizers of the points  $y_n$  have infinite index, then the stabilizer of the point  $y$  also has infinite index. This follows from the conclusion of [Lemma 3.3.2](#) and the fact that a finite index subgroup of a finitely generated group is also finitely generated.  $\square$

**Problem 3.3.3.** *Is it true that for an arbitrary infinite countable group  $\Gamma$ ,  $E^{ap}(\Gamma, 2^{\mathbb{N}})$  admits a compact action realization?*

We next note the following fact, which can be used to provide more examples of CBER that admit compact action realizations.

**Proposition 3.3.4.** *Let  $F$  be an aperiodic CBER on a standard Borel space  $X$ . Let  $Z \subseteq X$  be a Borel invariant set and put  $Y = X \setminus Z$  and  $E = F|Y$ . If  $E$  is not smooth and  $F|Z$  is hyperfinite, compressible, then  $E \cong_B F$ . So if  $F$  has a compact action realization, so does  $E$ .*

*Proof.* If  $F|Z$  is smooth, then  $F|Z \cong_B \mathbb{R}I_{\mathbb{N}}$  is Borel isomorphic to a direct sum of copies of  $I_{\mathbb{N}}$ , while if it is not smooth  $F|Z \cong_B E_t$ . Thus, by the Glimm-Effros Dichotomy and [Kec21b, 7.3 and 2.27] in the second case, we can find a decomposition  $Y = Y_0 \sqcup Y_1 \sqcup Y_2 \sqcup \dots \sqcup Y_{\infty}$  into invariant Borel sets such that  $F|Z \cong_B F|Y_n, \forall n \in \mathbb{N}$ . Let  $\pi_0$  be a Borel isomorphism of  $F|Z$  with  $F|Y_0$  and for  $n > 0$ , let  $\pi_n$  be a Borel isomorphism of  $F|Y_{n-1}$  with  $F|Y_n$ . Finally let  $\pi_{\infty}$  be the identity on  $Y_{\infty}$ . Then  $\bigcup_{n \in \mathbb{N}} \pi_n \cup \pi_{\infty}$  is a Borel isomorphism of  $F$  and  $E$ .  $\square$

**Corollary 3.3.5.** *Let  $F$  be an aperiodic CBER on a Polish space  $X$ . Then there is meager, invariant Borel set  $M \subseteq X$  such that for any invariant Borel set  $Y \supseteq M$ , if  $E = F|Y$  is not smooth, then  $E \cong_B F$ .*

*Proof.* By [KM04, 12.1 and 13.3], there is an invariant, comeager Borel set  $C \subseteq X$  such that  $F|C$  is compressible, hyperfinite. Put  $M = X \setminus C$ . If  $Y \supseteq M$  is invariant Borel such that  $E = F|Y$  is not smooth and  $Z = X \setminus Y$ , then we can apply Proposition 3.3.4.  $\square$

For example, let  $\Gamma$  be a countable group and consider a continuous, topologically transitive action of  $\Gamma$  on a compact Polish space  $X$  with infinite orbits. Then there is an invariant dense  $G_{\delta}$  set  $C \subseteq X$  consisting of points with dense orbits in  $X$  and such that if  $F$  is the equivalence relation induced by the action, then  $F|C$  is compressible, hyperfinite and non-smooth (as the action of  $\Gamma$  on  $C$  is topologically transitive). So  $F|C \cong_B E_t \cong_B \mathbb{R}E_t$ . Then by the countable chain condition for category, some copy of  $E_t$  in  $E|C$  is meager, so can subtract it from  $C$  and assume that if  $M = X \setminus C$ , then  $F|M$  is not smooth. It follows that for any invariant Borel set  $Y \supseteq M$ , if  $E = F|Y$ , then  $E \cong_B F$ , so that  $E$  has a compact action realization

Since for every  $E \in \mathcal{AE}$  on a Polish space  $X$  there is an invariant comeager Borel set  $Y \subseteq X$  such that  $E|Y$  is hyperfinite, it follows that if  $E \in \mathcal{AE}$  is not smooth when restricted to any invariant comeager Borel set, then there is an invariant comeager Borel set  $Y \subseteq X$  such that  $E|Y$  admits a minimal, compact action realization. Whether this holds for measure instead of category is an open problem.

**Problem 3.3.6.** *Let  $E \in \mathcal{AE}$  be on a standard Borel space  $X$  and let  $\mu$  be a measure on  $X$  such that the restriction of  $E$  to any invariant Borel set of measure 1 is not smooth. Is there is an invariant Borel set  $Y \subseteq X$  with  $\mu(Y) = 1$  such that  $E|_Y$  admits a compact action realization?*

(B) We next describe a “gluing” construction of two continuous actions of groups on compact Polish spaces at an orbit of one of the actions. We thank Aristotelis Panagiotopoulos for a useful discussion on this construction.

Let the countable group  $\Gamma$  act continuously on the compact Polish space  $X$  and let  $X_0 \subseteq X$  be an infinite orbit of this action. Let also the countable group  $\Delta$  act continuously on the compact Polish space  $Y$  with a fixed point  $y_0 \in Y$ . Fix compatible metrics  $d_X \leq 1$  and  $d_Y \leq 1$  for  $X$  and  $Y$ , respectively. Fix also a map  $x \mapsto |x|$  from  $X_0$  to  $\mathbb{R}^+$  such that  $\lim_{x \rightarrow \infty} |x| = +\infty$ , i.e., for every  $M \in \mathbb{R}^+$ , there if finite  $F \subseteq X_0$  such that  $x \notin F \implies |x| > M$ . For each  $x \in X_0$ , let  $Y_x$  be a set and let  $\pi_x$  be a bijection  $\pi_x: Y \rightarrow Y_x$  such that  $\pi_x(y_0) = x$  and  $x_1 \neq x_2 \implies Y_{x_1} \cap Y_{x_2} = \emptyset$ . Put  $Y'_x = Y_x \setminus \{x\}$  and let  $Z = X \sqcup \bigsqcup_{x \in X_0} Y'_x$ . Define a metric  $d_x$  on  $Y_x$  as follows:

$$d_x(y_1, y_2) = \frac{d_Y(\pi_x^{-1}(y_1), \pi_x^{-1}(y_2))}{|x|}.$$

Then define a metric  $d_Z$  on  $Z$  as follows:

$$\begin{aligned} d_Z(x_1, x_2) &= d_X(x_1, x_2), \text{ if } x_1, x_2 \in X, \\ d_Z(y_1, y_2) &= d_x(y_1, y_2), \text{ if } y_1, y_2 \in Y_x, x \in X_0, \\ d_Z(y, x') &= d_x(y, x) + d_X(x, x'), \text{ if } y \in Y_x, x \in X_0, x' \in X, \\ d_Z(y_1, y_2) &= d_{x_1}(y_1, x_1) + d_X(x_1, x_2) + d_{x_2}(x_2, y_2), \text{ if } y_1 \in Y_{x_1}, y_2 \in Y_{x_2}, x_1 \neq x_2. \end{aligned}$$

**Remark 3.3.7.** We note here that in the preceding “gluing” construction, if the spaces  $X, Y$  are 0-dimensional, so is the space  $Z$ . To see this we start with metrics  $d_X, d_Y$  as above which are actually ultrametrics (these exist since  $X, Y$  are 0-dimensional). Then it is enough to show that for every  $z \in Z$ , there is an  $\varepsilon_z > 0$  such that every open ball (in the metric  $d_Z$ )  $B_z(\varepsilon)$ , for  $\varepsilon < \varepsilon_z$ , is closed. Below recall that open balls in ultrametrics are closed.

Consider first the case where  $z \in X$  and fix  $z_1, z_2, \dots \in B_z(\varepsilon)$  with  $z_n \rightarrow z_\infty$ . If infinitely many  $z_n$  are in  $X$ , then clearly  $z_\infty \in B_z(\varepsilon)$  as  $d_X$  is an ultrametric. Otherwise, we can assume that all  $z_n$  are in  $Z \setminus X$ . If now there is some  $x \in X_0$  such that infinitely many  $z_n \in Y'_x$ , so that  $z_\infty \in Y_x$ , we have  $d_Z(z_n, z) = d_x(z_n, x) + d_X(x, z)$ , so  $d_x(z_n, x) < \varepsilon - d_X(x, z)$ , thus, since  $d_x$  is an ultrametric,  $d_x(z_\infty, x) <$

$\varepsilon - d_X(x, z)$  and thus  $d_Z(z_\infty, z) < \varepsilon$ . Otherwise there is a subsequence  $(z_{n_i})$  and  $x_i \in X_0$  with  $z_{n_i} \in Y'_{x_i}$  and  $x_i$  converges to  $x \in X$  and thus  $z_{n_i} \rightarrow z_\infty = x$  (since  $d_Z(z_{n_i}, x_i) < \frac{1}{|x_i|}$ ). Now  $d_Z(z, z_{n_i}) = d_X(z, x_i) + d_{x_i}(x_i, z_{n_i}) < \varepsilon$ , so  $d_X(z, x_i) < \varepsilon$  and, since  $d_X$  is an ultrametric,  $d_Z(z, z_\infty) = d_X(z, z_\infty) < \varepsilon$ .

The other case is when  $z \in Y'_x$ , for some  $x \in X_0$ . Take  $\varepsilon_z = d_x(z, x)$ . Then for  $\varepsilon < \varepsilon_z$  the open ball  $B_z(\varepsilon)$  is the same as the open ball of radius  $\varepsilon$  in the metric  $d_x$ , so the proof is complete.

**Proposition 3.3.8.**  *$(Z, d_Z)$  is a compact metric space.*

*Proof.* It is routine to check that  $d_Z$  is a metric on  $Z$ . We next verify compactness. Let  $(z_n)$  be a sequence in  $Z$  in order to find a converging subsequence. The other cases been obvious, we can assume that  $z_n \in Y_{x_n}$  with  $x_n \in X_0$  distinct and therefore  $|x_n| \rightarrow \infty$ , in which case, by going to a subsequence, we can also assume that  $x_n \rightarrow x \in X$ . Since  $d_Z(z_n, x_n) \leq \frac{1}{|x_n|}$ , it follows that  $z_n \rightarrow x$ .  $\square$

We next define an action of  $\Delta$  on  $Z$ . Given  $\delta \in \Delta$  and  $z \in Z$  we define  $\delta \cdot z$  as follows:

$$\begin{aligned} \delta \cdot z &= \pi_x(\delta \cdot \pi_x^{-1}(z)), \text{ if } z \in Y_x, x \in X_0, \\ \delta \cdot z &= z, \text{ if } z \in X. \end{aligned}$$

If we identify each  $Y_x$  with  $Y$ , then this action “extends” the action of  $\Delta$  on  $Y$ .

We finally extend the action of  $\Gamma$  from  $X$  to all of  $Z$ . Given  $\gamma \in \Gamma$  and  $z \in Z$  define  $\gamma \cdot z$  as follows:

$$\begin{aligned} \gamma \cdot z &= z, \text{ if } z \in X, \\ \gamma \cdot z &= \pi_{\gamma \cdot x}(\pi_x^{-1}(z)), \text{ if } z \in Y_x, x \in X_0. \end{aligned}$$

It is easy to see that these two actions commute, so they give an action of  $\Gamma \times \Delta$  on  $Z$ .

**Proposition 3.3.9.** *The action of  $\Gamma \times \Delta$  on  $Z$  is continuous.*

*Proof.* It is enough to check that the action of  $\Gamma$  on  $Z$  is continuous and so is the action of  $\Delta$ .

Let first  $\gamma \in \Gamma$  and  $z_n \in Z$  be such that  $z_n \rightarrow z$ , in order to show that  $\gamma \cdot z_n \rightarrow \gamma \cdot z$ . It is enough to find a subsequence  $(n_i)$  such that  $\gamma \cdot z_{n_i} \rightarrow \gamma \cdot z$ . Again, the other cases being trivial, we can assume that  $z_n \in Y_{x_n}$  with  $x_n \in X_0$  distinct, so that also  $|x_n| \rightarrow \infty$ , in which case, by going to a subsequence, we can also assume that  $x_n \rightarrow x \in X$ . Then  $\gamma \cdot x_n \rightarrow \gamma \cdot x$  and  $d_Z(\gamma \cdot z_n, \gamma \cdot x_n) \leq \frac{1}{|\gamma \cdot x_n|} \rightarrow 0$ , as the  $\gamma \cdot x_n$  are



also distinct and thus  $|\gamma \cdot x_n| \rightarrow \infty$ . Since  $d_Z(z_n, x_n) \leq \frac{1}{|x_n|}$ , clearly  $x = z$ , and thus  $\gamma \cdot z_n \rightarrow \gamma \cdot z$ .

Let now  $\delta \in \Delta$  and  $z_n \in Z$  be such that  $z_n \rightarrow z$ , in order to show that  $\delta \cdot z_n \rightarrow \delta \cdot z$ . It is enough again to find a subsequence  $(n_i)$  such that  $\delta \cdot z_{n_i} \rightarrow \delta \cdot z$  and as before we can assume that  $z_n \in Y_{x_n}$  with  $x_n \in X_0$  distinct, so that also  $|x_n| \rightarrow \infty$ , in which case, by going to a subsequence, we can also assume that  $x_n \rightarrow x \in X$ . Then  $\delta \cdot x_n = x_n \rightarrow \delta \cdot x = x$ . Now  $\delta \cdot z_n \in Y_{x_n}$ , so that  $\delta_Z(\delta \cdot z_n, x_n) \rightarrow 0$  and  $d_Z(z_n, x_n) \rightarrow 0$ . Thus  $z = x$  and  $\delta \cdot z_n \rightarrow \delta \cdot z = z$ .  $\square$

Let now  $E$  be the equivalence relation induced by the action of  $\Gamma$  on  $X$ , let  $F$  be the equivalence relation induced by the action of  $\Delta$  on  $Y \setminus \{y_0\}$  and finally let  $G$  be the equivalence relation induced by the action of  $\Gamma \times \Delta$  on  $Z$ . Then it is easy to check the following;

**Proposition 3.3.10.**  $G \cong_B E \oplus (F \times I_{\mathbb{N}})$

We present now an application of this construction to the problem of compact action realizations.

**Theorem 3.3.11.** *Let the CBER  $F$  be induced by a continuous action of a countable group on a locally compact Polish space. Then  $F \times I_{\mathbb{N}}$  admits a compact action realization. In particular, if  $F$  is compressible,  $F$  admits a compact action realization.*

*Moreover, if the locally compact space is 0-dimensional,  $F \times I_{\mathbb{N}}$  admits a compact action realization on the Cantor space  $2^{\mathbb{N}}$ .*

*Proof.* In the preceding “gluing” construction, take  $X = 2^{\mathbb{N}}$  and a continuous action of  $\Gamma = \mathbb{F}_2$  such that  $E = E_t$ . Fix also a countable group  $\Delta$  and a continuous action of  $\Delta$  on a locally compact space  $Y'$  which induces  $F$ . Let  $Y = Y' \sqcup \{y_0\}$  be the one-point compactification of  $Y'$  (if  $Y'$  is already compact, we obtain  $Y$  by adding an isolated point to  $Y'$ ). Then the action of  $\Delta$  can be continuously extended to  $Y$  by fixing  $y_0$ . Thus we have by [Proposition 3.3.10](#) that  $E \oplus (F \times I_{\mathbb{N}})$  admits a compact action realization. Since  $F$  is not smooth, we have, as in the proof of [Proposition 3.3.4](#), that  $E \oplus (F \times I_{\mathbb{N}}) \cong_B (F \times I_{\mathbb{N}})$  and the proof is complete.

In the case that  $Y'$  is 0-dimensional, by [Remark 3.3.7](#)  $F \times I_{\mathbb{N}}$  admits a compact action realization on a 0-dimensional space  $Z$ . By going to the perfect kernel of  $Z$ , we can assume that  $Z$  is perfect (see the proof of [Proposition 3.2.1](#)), thus homeomorphic to the Cantor space.  $\square$

For a sequence  $(E_n)$  of CBER, we let  $\bigoplus_n E_n$  be the direct sum of this sequence. If  $E_n$  is on the space  $X_n$ , then  $E = \bigoplus_n E_n$  is the equivalence relation on the space  $\bigsqcup_n X_n$ , where  $xEy \iff \exists n(x, y \in X_n \text{ and } xE_n y)$ . The following is an immediate consequence of [Theorem 3.3.11](#).

**Corollary 3.3.12.** *Let each  $E_n \in \mathcal{AE}$  admit a compact action realization. Then  $\bigoplus_n E_n \times I_{\mathbb{N}}$  also admits a compact action realization. In particular, if also every  $E_n$  is compressible,  $\bigoplus_n E_n$  admits a compact action realization.*

Recall that a CBER  $E$  is **universal** if for every CBER  $F$  we have  $F \leq_B E$ . Note that by [CK18, Proposition 3.27 (ii)], there is a unique, up to Borel isomorphism, compressible, universal CBER. The following are immediate consequences of Theorem 3.3.11.

**Corollary 3.3.13.** *Let  $E$  be a compressible, universal CBER. Then  $E$  admits a transitive, compact action realization on the Cantor space  $2^{\mathbb{N}}$ .*

*Proof.* Let us first note that there exists a compressible, universal CBER  $F$  that is generated by a continuous action of a countable group on  $2^{\mathbb{N}}$ . Indeed, let  $E(\mathbb{F}_2, 2)$  be the equivalence relation generated by the canonical action of  $\mathbb{F}_2$  on  $2^{\mathbb{F}_2}$ . Consider the equivalence relation  $F = E(\mathbb{F}_2, 2) \times I_{\mathbb{N}}$ . This equivalence relation is compressible, universal. By Theorem 3.3.11,  $F$  has a continuous action realization on  $2^{\mathbb{N}}$ . An inspection of the “gluing” construction involved in the proof of Theorem 3.3.11 shows that this action is topologically transitive.  $\square$

**Corollary 3.3.14.** *Let  $E$  be a compressible, universal CBER. Then  $E$  admits a minimal action realization on the Baire space  $\mathbb{N}^{\mathbb{N}}$ .*

*Proof.* By Corollary 3.3.13 consider a continuous action  $a$  of a countable group  $G$  on  $2^{\mathbb{N}}$ , which induces an equivalence relation  $F$  which is Borel isomorphic to  $E$ . Then consider the Borel map  $f$  that sends  $x \in 2^{\mathbb{N}}$  to the closure of its orbit (which is a member of the space of all compact subsets of  $2^{\mathbb{N}}$ ), By [MSS16, Theorem 3.1], there is some  $K$  such that  $F|f^{-1}(K)$  is universal. But clearly  $Z = f^{-1}(K)$  is a  $G_{\delta}$  set, so a Polish, 0-dimensional space, invariant under the action  $a$ . Moreover this action restricted to  $Z$  is minimal. As in the proof of Proposition 3.2.1, we can find a subspace  $Y$  of  $Z$  homeomorphic to  $\mathbb{N}^{\mathbb{N}}$  invariant under the action, such that  $F|Z \cong_B F|Y$ . Thus  $F|Z$  is induced by a minimal action on the Baire space and is compressible, universal. As in the proof of Corollary 3.3.13, this shows that every compressible, universal CBER admits a minimal action realization on the Baire space.  $\square$

The following is an open problem:

**Problem 3.3.15.** *Does an arbitrary (not necessarily compressible) aperiodic, universal CBER admit a compact action realization?*

In the next Section 3.6 we will consider realizations of equivalence relations by subshifts and in particular prove a considerable strengthening of Corollary 3.3.13.

### 3.4 Turing and arithmetical equivalence

Below let  $\equiv_T$  denote **Turing equivalence** and  $\equiv_A$  **arithmetical equivalence**.

The following is an immediate consequence of [Corollary 3.3.13](#), since  $\equiv_A$  is compressible and universal by [\[MSS16\]](#):

**Corollary 3.4.1.** *Arithmetical equivalence  $\equiv_A$  on  $2^{\mathbb{N}}$  admits a compact action realization on  $2^{\mathbb{N}}$ , which is in fact a minimal subshift of  $2^{\mathbb{F}_4}$ .*

On the other hand the following is open:

**Problem 3.4.2.** *Does Turing equivalence  $\equiv_T$  on  $2^{\mathbb{N}}$  admit a compact action realization?*

A negative answer to this question will on the one hand provide a new proof of the non-hyperfiniteness of  $\equiv_T$  but, more importantly, give a negative answer to the long-standing problem of the universality of  $\equiv_T$ , see [\[DK00\]](#).

Concerning Turing equivalence, we know from [Proposition 3.2.1](#) that it admits a continuous action realization on the Baire space  $\mathbb{N}^{\mathbb{N}}$ , i.e., that there is a Borel isomorphism of  $2^{\mathbb{N}}$  with  $\mathbb{N}^{\mathbb{N}}$  which sends  $\equiv_T$  to an equivalence relation induced by a continuous action of a countable group on  $\mathbb{N}^{\mathbb{N}}$ . We calculate below an upper bound for the Baire class of such a Borel isomorphism. A version of the next theorem was first proved by Andrew Marks, in response to an inquiry of the authors, with “Baire class 3” instead of “Baire class 2”. The proof of [Theorem 3.4.3](#) below uses some of his ideas along with other additional arguments.

**Theorem 3.4.3.** *There exists a Baire class 2 map  $\Phi: 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  that is an isomorphism between  $\equiv_T$  and an equivalence relation given by a continuous group action on  $\mathbb{N}^{\mathbb{N}}$ .*

The most natural construction of the isomorphism will yield [Proposition 3.4.4](#) below. We will show later that it in fact implies [Theorem 3.4.3](#).

**Proposition 3.4.4.** *There exists a Baire class 2 map  $\Psi$  that is an isomorphism between  $\equiv_T$  on  $2^{\mathbb{N}}$  and an equivalence relation given by a continuous group action on a 0-dimensional Polish space.*

*Proof.* Let  $\varphi^i$  denote the partial function computed by the  $i$ th Turing machine, in some recursive enumeration of all the Turing machines, such that  $\varphi^0$  is the identity on  $2^{\mathbb{N}}$ . That is, we consider Turing machines with oracle and input tapes, and  $\varphi^i(x) = y$  iff for each  $n$  the  $i$ th Turing machine with oracle  $x$  and input  $n$  halts with the output  $y(n)$ .

We start with an easy observation. Below, for  $s \in 2^{<\mathbb{N}}$ , put  $[s] = \{x \in 2^{\mathbb{N}} : s \subseteq x\}$ .

**Lemma 3.4.5.** *Assume that  $x \equiv_T y$ . There exists an  $i$  with  $\varphi^i(x) = y$  and  $\varphi^i(y) = x$ .*

*Proof.* We can assume that  $x \neq y$ . Pick  $j, k \in \mathbb{N}$  with  $\varphi^j(x) = y$  and  $\varphi^k(y) = x$ , and  $n$  with  $x \upharpoonright n \neq y \upharpoonright n$ . Then, an  $i$  with  $\varphi^i \upharpoonright [x \upharpoonright n] = \varphi^j \upharpoonright [x \upharpoonright n]$  and  $\varphi^i \upharpoonright [y \upharpoonright n] = \varphi^k \upharpoonright [y \upharpoonright n]$  clearly works.  $\square$

The idea is to define a coding function  $\Psi = (\alpha, \beta, \gamma)$  that will serve as an isomorphism. The crucial property of  $\Psi(x)$  is that  $\alpha$  encodes for each  $i$  whether  $\varphi^i$  is an involution on  $x$  (and does this for every  $y \equiv_T x$ ),  $\beta$  will ensure that  $\Psi$  is continuous, while  $\gamma$  will be used to code  $x$  and its  $\equiv_T$  equivalence class.

Let us now give the precise definitions. Fix a function  $\iota : \mathbb{N}^2 \rightarrow \mathbb{N}$  such that for each  $i, j \in \mathbb{N}$  we have  $\varphi^{\iota(i,j)} = \varphi^i \circ \varphi^j$ .

Let  $\beta : 2^{\mathbb{N}} \rightarrow (\mathbb{N} \cup \{*\})^{\mathbb{N}^3}$  be defined by  $\beta(x)(i, j, m) = n$ , if  $n$  is least such that both the  $i$ th and the  $j$ th Turing machines with oracle  $x$  and input  $m$  halt with the same output in at most  $n$  steps, and let  $\beta(x)(i, j, m) = *$ , if such an  $n$  does not exist.

Define a map  $\alpha : 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}^2}$  by letting

$$\alpha(x)(i, j) = 0 \iff \beta(x)(i, j) \in \mathbb{N}^{\mathbb{N}},$$

and

$$\alpha(x)(i, j) = m + 1 \iff m \text{ is least with } \beta(x)(i, j, m) = *.$$

Let  $\gamma : 2^{\mathbb{N}} \rightarrow (2 \cup \{*\})^{\mathbb{N}^2}$  be defined by

$$\gamma(x)(i, k) = *, \text{ if } \alpha(x)(\iota(i, i), 0) \neq 0,$$

and

$$\gamma(x)(i, k) = \varphi^i(x)(k),$$

otherwise.

Note that by the choice of  $\varphi^0$  for each  $x \in 2^{\mathbb{N}}$  we have that  $\gamma(x)(0) = x$ .

Finally, let  $\Psi(x) = (\alpha(x), \beta(x), \gamma(x))$ . Let us denote the space  $\mathbb{N}^{\mathbb{N}^2} \times (\mathbb{N} \cup \{*\})^{\mathbb{N}^3} \times (2 \cup \{*\})^{\mathbb{N}^2}$  by  $X$ , where  $\mathbb{N} \cup \{*\}$  and  $2 \cup \{*\}$  are endowed with the discrete topology.

**Lemma 3.4.6.**  *$\Psi(2^{\mathbb{N}})$  is closed in  $X$ .*

*Proof.* Assume that  $(\alpha(x_k), \beta(x_k), \gamma(x_k))_k$  is a convergent sequence. It follows from the choice of  $\varphi^0$ , the definition of  $\gamma$ , and  $\gamma(x_k)(0) \rightarrow \gamma(x)(0)$  that  $x_k \rightarrow x$  holds. Take any  $i, j, m \in \mathbb{N}$ . It is clear from the definition of  $\beta$  that  $\beta(x)(i, j, m) = n$  holds for some  $n \in \mathbb{N}$  if and only if  $\beta(x_k)(i, j, m) = n$  is true for every large enough  $k$ . This shows that  $\beta(x_k) \rightarrow \beta(x)$ .

Using this, it is easy to check that  $\alpha(x_k) \rightarrow \alpha(x)$  holds as well.

Finally, by  $\alpha(x_k) \rightarrow \alpha(x)$ , we have that  $\alpha(x_k)(i, j) = \alpha(x)(i, j)$  for each large enough  $k$ . This of course implies  $\gamma(x_k) \rightarrow \gamma(x)$  by the continuity of the functions  $\varphi^i$ .  $\square$

Since  $\varphi^0$  is the identity and by the definition of  $\gamma$ , we have that  $\gamma(x)(0) = x$  for every  $x$ . In particular,  $\Psi$  is injective.

For  $i \in \mathbb{N}$  define a map  $\delta_i$  from  $X$  to itself as follows:

$$\delta_i(\Psi(x)) = \Psi(x) \text{ if } \alpha(x)(\iota(i, i), 0) \neq 0,$$

otherwise

$$\delta_i(\Psi(x)) = \Psi(\varphi^i(x)).$$

**Lemma 3.4.7.** *The maps  $(\delta_i)_{i \in \mathbb{N}}$  are  $\Psi(2^{\mathbb{N}}) \rightarrow \Psi(2^{\mathbb{N}})$  homeomorphisms.*

*Proof.* Fix  $i \in \mathbb{N}$ . It is easy to check that on the set  $\{\Psi(x) : \alpha(x)(\iota(i, i), 0) = 0\}$  for each  $i', j', m \in \mathbb{N}$  we have that:

$$\delta_i(\alpha(x), \beta(x), \gamma(x))(0)(i', j') = (\alpha(x))(\iota(i', i), \iota(j', i)),$$

$$\delta_i(\alpha(x), \beta(x), \gamma(x))(1)(i', j', m) = (\beta(x))(\iota(i', i), \iota(j', i), m),$$

and

$$\delta_i(\alpha(x), \beta(x), \gamma(x))(2)(i', m) = (\gamma(x))(i, m), \text{ if } \alpha(\iota(\iota(i', i'), i), i) \neq 0,$$

while

$$\delta_i(\alpha(x), \beta(x), \gamma(x))(2)(i', m) = (\gamma(x))(\iota(i', i), m), \text{ otherwise.}$$

As  $\delta_i$  is equal to identity on a relatively clopen set, while it selects and permutes some of the coordinates on its complement, it follows that  $\delta_i$  is continuous.

Finally, we show that  $\delta_i(\delta_i(\Psi(x))) = \Psi(x)$  holds for each  $x$ . Indeed, the set  $\{x : \alpha(x)(\iota(i, i), 0) = 0\}$  is the collection of binary sequences on which  $\varphi^i$  is an involution, so it follows from the definition of  $\delta_i$  that on the  $\Psi$  image of this set our lemma holds. Moreover, on the complement of this set,  $\delta_i$  is the identity, which finishes the proof of the lemma.  $\square$

Let  $E_\Delta$  be the equivalence relation on  $\Psi(2^\omega)$  generated by the maps  $\{\delta_i : i \in \mathbb{N}\}$ .

**Lemma 3.4.8.**  *$\Psi$  is an isomorphism between  $\equiv_T$  and  $E_\Delta$ .*

*Proof.* First, it is clear from the definition of  $\delta_i$  that  $\delta_i(\Psi(x)) = \Psi(y)$  implies that  $x \equiv_T y$ . So  $\Psi^{-1}$  is a homomorphism.

Second, assume that  $x \equiv_T y$ . Then by Lemma 3.4.5 there exists an  $i$  with  $\varphi^i(x) = y$  and  $\varphi^i(y) = x$ . Then  $\alpha(x)(i, i, 0) = 0$ , so  $\delta_i(\Psi(x)) = \Psi(\varphi^i(x)) = \Psi(y)$ , so  $\Psi(x)E_\Delta\Psi(y)$ .  $\square$

Now we turn to the calculation of the complexity of the map  $\Psi$ .

**Lemma 3.4.9.** *The map  $\beta$  is Baire class 1 and the maps  $\alpha$  and  $\gamma$  are Baire class 2. Consequently, the map  $\Psi$  is Baire class 2.*

*Proof.* For  $\beta$ , take any  $i, j, m \in \mathbb{N}$ . Then for each natural number  $n$ , the set  $\{x : \beta(x)(i, j, m) = n\}$  is open. Thus, the set  $\{x : \beta(x)(i, j, m) = *\}$  is closed. This shows that  $\beta$  preimages of basic clopen sets are  $\Delta_2^0$ .

For  $\alpha$ , for a given  $i$ , the set  $\{x : \alpha(x)(i, j) = 0\} = \{x : \forall m(\beta(x)(i, j, m) \in \omega)\}$  is  $\Pi_2^0$ , and also, for  $m \neq 0$  we have that  $\{x : \alpha(x)(i, j) = m\} = \{x : \forall m' < m(\beta(x)(i, j, m') \in \mathbb{N} \wedge \beta(x)(i, j, m) = *)\}$ , which shows that these sets are  $\Pi_2^0$  as well, and thus  $\alpha$  is indeed Baire class 2. Finally, just note that  $\gamma$  depends continuously on  $\alpha$ , so it must be Baire class 2.  $\square$

This completes the proof of Proposition 3.4.4  $\square$

In order to finish the proof of Theorem 3.4.3 we need a last observation.

**Lemma 3.4.10.** *Assume that  $\Gamma$  acts continuously on an uncountable and zero-dimensional Polish space  $X$ , so that the induced equivalence relation  $E_\Gamma^X$  is aperiodic. Then there exist invariant under the action  $X' \subseteq X$  that is homeomorphic to  $\mathbb{N}^\mathbb{N}$ , and an isomorphism  $\varphi$  between  $E_\Gamma^X$  and  $E_\Gamma^X \upharpoonright X'$  that moves only countably many points.*

*Proof.* As in the proof of Proposition 3.2.1.  $\square$

*Proof of Theorem 3.4.3.* By Proposition 3.4.4 there exists a Baire class 2 isomorphism between  $\equiv_T$  and some equivalence relation of the form  $E_\Gamma^X$ , where  $\Gamma$  acts continuously on a zero-dimensional Polish space  $X$ . Applying Lemma 3.4.10 we get an isomorphism with an equivalence relation on the Baire space. Moreover, as countable modifications of Baire class 2 functions do not change their class, we are done.  $\square$

We do not know if the complexity of the Borel isomorphism in Theorem 3.4.3 is optimal.

**Problem 3.4.11.** *Is there a Baire class 1 map that is an isomorphism between  $\equiv_T$  and an equivalence relation given by a continuous group action on  $\mathbb{N}^{\mathbb{N}}$ ?*

Consider now *any* map  $\Phi: 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  satisfying the conditions of [Theorem 3.4.3](#). Then for some  $p \in \mathbb{N}^{\mathbb{N}}$  we have  $x \equiv_T y \implies \langle \Phi(x), p \rangle \equiv_T \langle \Phi(y), p \rangle$ . Thus if Martin's Conjecture is true, we have that on a cone  $\langle \Phi(x), p \rangle$  is Turing equivalent to one of  $x, x', x''$  and thus the same is true for  $\Phi(x)$ . For the *particular*  $\Phi$  that was constructed in the proof of [Theorem 3.4.3](#), it is easy to see that  $\Phi(x) \equiv_T x''$ , since  $x''$  can be easily computed from the map  $\alpha$  defined in the proof of [Proposition 3.4.4](#). Similar to the problem [Problem 3.4.11](#), we have the following:

**Problem 3.4.12.** *Is there a Borel map  $\Phi: 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  that is an isomorphism between  $\equiv_T$  and an equivalence relation given by a continuous group action on  $\mathbb{N}^{\mathbb{N}}$  such that  $\Phi(x) \equiv_T x'$  on a cone?*

On the other hand we have the following result, where for  $x, y \in \mathbb{N}^{\mathbb{N}}$ ,  $x \leq_T y$  iff  $x$  is recursive in  $y$ .

**Proposition 3.4.13.** *There is no Borel map  $\Phi: 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  that is an isomorphism between  $\equiv_T$  and an equivalence relation given by a continuous group action on  $\mathbb{N}^{\mathbb{N}}$  such that  $\Phi(x) \leq_T x$  on a cone.*

*Proof.* Recall that a **pointed perfect tree** is a perfect binary tree  $S \subseteq 2^{<\mathbb{N}}$  such that  $x \in [S] \implies S \leq_T x$ , where  $[S] \subseteq 2^{\mathbb{N}}$  is the set of infinite branches of  $S$ . Below we will use certain properties of pointed perfect trees due to Martin, whose proofs can be found, for example, in [\[Kec88\]](#). Assume that  $\Phi(x) \leq_T x$  on a cone, towards a contradiction. Then by [\[Kec88, Theorem 1.3\]](#) there is a perfect pointed tree  $T$  such that  $x \in [T] \implies \Phi(x) \leq_T x$ . Then by [\[Kec88, Lemma 1.4\]](#), there is a perfect pointed subtree  $S \subseteq T$  and  $i \in \mathbb{N}$  such that if  $x \in [S]$ , then  $\varphi^i(x)$  is defined and  $\varphi^i(x) = \Phi(x)$ . Thus  $\Phi$  is continuous on  $[S]$ . It follows that  $\equiv_T$  restricted to  $[S]$  is  $\Sigma_2^0$ . Let now  $\Psi$  be the canonical homeomorphism of  $2^{\mathbb{N}}$  with  $[S]$ , so that  $\Psi(x) \equiv_T x$ , if  $S \leq_T x$ . It follows that for  $S \leq_T x, y$ , we have  $x \equiv_T y \iff \Psi(x) \equiv_T \Psi(y)$ , thus, in particular, for some  $z \in 2^{\mathbb{N}}$ , the Turing degree of  $z$ , i.e., the set  $\{w \in 2^{\mathbb{N}} : w \equiv_T z\}$  is  $\Sigma_2^0(z)$ . This is false in view of the following well-known fact:

**Lemma 3.4.14.** *For any  $z \in 2^{\mathbb{N}}$ , the Turing degree of  $z$  is in  $\Sigma_3^0(z)$  but not in  $\Pi_3^0(z)$ .*

*Proof.* It is easy to check that the Turing degree of  $z$  is  $\Sigma_3^0(z)$ . Assume now that it is in  $\Pi_3^0(z)$ , towards a contradiction. Then if  $A = \{w \in 2^{\mathbb{N}} : w \leq_T z\}$ , we have that  $A$  is also  $\Pi_3^0(z)$ , since  $w \in A \iff \langle w, z \rangle \equiv_T z$ . But then  $2^{\mathbb{N}} \setminus A$  is a comeager  $\Sigma_3^0(z)$  set, so by the relativized version of the basis theorem of Shoenfield [\[Sho58\]](#) it contains a recursive in  $z$  real, a contradiction.  $\square$

□

### 3.5 Continuous actions on compact spaces, compressibility and paradoxicality

(A) In connection with [Problem 3.2.10](#), for the case of compact action realizations, we discuss some special properties of continuous actions of countable groups on compact Polish spaces that may have some relevance to this question.

Let  $\Gamma$  be a countable group and let  $\mathbf{a}$  be a Borel action of  $\Gamma$  on a standard Borel space  $X$ . Put  $\gamma \cdot x = \mathbf{a}(\gamma, x)$ . We denote by  $\langle \mathbf{a} \rangle$  the set of all Borel maps  $T: X \rightarrow X$  such that  $\forall x \exists \gamma \in \Gamma (T(x) = \gamma \cdot x)$ . Equivalently this means that there is a Borel partition  $X = \bigsqcup_{\gamma \in \Gamma} X_\gamma$  such that  $T(x) = \gamma \cdot x$  for  $x \in X_\gamma$ . We also let  $\langle \mathbf{a} \rangle^f$  consist of all Borel maps  $T: X \rightarrow X$  for which there is a *finite* subset  $F \subseteq \Gamma$  such that  $\forall x \exists \gamma \in F (T(x) = \gamma \cdot x)$ . Equivalently this means that there is a Borel partition  $X = \bigsqcup_{\gamma \in F} X_\gamma$  such that  $T(x) = \gamma \cdot x$  for  $x \in X_\gamma$ .

We say that the action  $\mathbf{a}$  is **compressible** (resp., **finitely compressible**) if there is an injective Borel map in  $T \in \langle \mathbf{a} \rangle$  (resp.,  $T \in \langle \mathbf{a} \rangle^f$ ) such that for every orbit  $C$  of  $\mathbf{a}$ ,  $T(C) \subsetneq C$  or equivalently  $\Gamma \cdot (X \setminus T(X)) = X$ . Clearly the action  $\mathbf{a}$  is compressible iff the associated equivalence relation is compressible. The action  $\mathbf{a}$  is **paradoxical** (resp., **finitely paradoxical**) if there are two injective Borel maps  $T_1, T_2$  in  $\langle \mathbf{a} \rangle$  (resp., in  $\langle \mathbf{a} \rangle^f$ ) such that  $T_1(X) \cap T_2(X) = \emptyset, T_1(X) \cup T_2(X) = X$ .

Clearly if  $\mathbf{a}$  is paradoxical (resp., finitely paradoxical), then  $\mathbf{a}$  is compressible (resp., finitely compressible). It is also known that if  $\mathbf{a}$  is compressible, then  $\mathbf{a}$  is paradoxical; see, e.g., [\[Kec21b, 2.23\]](#).

**Remark 3.5.1.** It is easy to see that finite compressibility does not imply finite paradoxicality. Take for example  $\mathbb{Z}$  acting on itself by translation. Since  $\mathbb{Z}$  is amenable this action is not finitely paradoxical. On the other hand the map  $T: \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $T(n) = n$ , if  $n < 0$ , and  $T(n) = n + 1$ , if  $n \geq 0$ , shows that this action is finitely compressible.

**Remark 3.5.2.** One can easily see that finite paradoxicality is equivalent to the following strengthening of finite compressibility: There is an injective Borel map  $T \in \langle \mathbf{a} \rangle^f$  and a finite subset  $F \subseteq \Gamma$  such that  $F \cdot (X \setminus T(X)) = X$ .

For  $n \geq 1$ , let  $[n] = \{1, 2, \dots, n\}$ . The  **$n$ -amplification** of  $\mathbf{a}$  is the action  $\mathbf{a}_n$  of the group  $\Gamma \times S_n$  on  $X \times [n]$  given by  $(\gamma, \pi) \cdot (x, i) = (\gamma \cdot x, \pi(i))$ , where  $S_n$  is the group of permutations of  $[n]$ . An **amplification** of  $\mathbf{a}$  is an  $n$ -amplification of  $\mathbf{a}$ , for some  $n$ .



**Theorem 3.5.3.** *Let  $\mathbf{a}$  be a continuous action of a countable group  $\Gamma$  on a compact Polish space  $X$ . Then the following are equivalent:*

- (i)  $\mathbf{a}$  is compressible;
- (ii)  $\mathbf{a}$  is paradoxical;
- (iii) an amplification of  $\mathbf{a}$  is finitely compressible;
- (iv) an amplification of  $\mathbf{a}$  is finitely paradoxical.

*Proof.* (1) The proof will be based on Nadkarni's Theorem and the following two results. We first recall some standard terminology:

Let  $X$  be a standard Borel space and let  $B(X)$  be the  $\sigma$ -algebra of its Borel sets. A **finitely additive Borel probability measure** is a map  $\mu: B(X) \rightarrow [0, 1]$  such that  $\mu(\emptyset) = 0$ ,  $\mu(X) = 1$ , and  $\mu(A \cup B) = \mu(A) + \mu(B)$ , if  $A \cap B = \emptyset$ . It is countably additive if moreover  $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ , for any pairwise disjoint family  $(A_n)$ . Recall that we call these simply *measures*. If  $\mathbf{a}$  is a Borel action of a countable group  $\Gamma$  on  $X$ , then  $\mu$  is invariant if for any Borel set  $A$  and  $\gamma \in \Gamma$ ,  $\mu(\gamma \cdot A) = \mu(A)$ .

**Theorem 3.5.4** ([Tse15, 5.3]). *Let  $\Gamma$  be a countable group and let  $\mathbf{a}$  be a continuous action of  $\Gamma$  on a compact Polish space  $X$ . If  $\mathbf{a}$  admits an invariant finitely additive Borel probability measure, then it admits an invariant measure.*

**Remark 3.5.5.** The hypothesis that  $X$  is compact Polish is necessary here. From Remark 3.5.9 we see that there is a counterexample to this statement even with  $X$  Polish locally compact.

**Theorem 3.5.6** ([TW16, 11.3]). *Let  $\Gamma$  be a countable group and let  $\mathbf{a}$  be a Borel action of  $\Gamma$  on a standard Borel space  $X$ . Then the following are equivalent:*

- (i) there is no invariant finitely additive Borel probability measure on  $X$ ;
- (ii) there is a finitely paradoxical amplification of  $\mathbf{a}$ .

(2) We now prove Theorem 3.5.3. We have already mentioned (in the paragraph preceding Remark 3.5.1) the equivalence of (i) and (ii).

(i)  $\implies$  (iv): If  $\mathbf{a}$  is compressible, then by Nadkarni's Theorem it does not admit an invariant measure, so by Theorem 3.5.4 it does not admit an invariant finitely additive Borel probability measure. Then by Theorem 3.5.6 some amplification of  $\mathbf{a}$  is finitely paradoxical.

(iv)  $\implies$  (iii) is obvious.

(iii)  $\implies$  (i): Assume that for some  $n$  the amplification  $\mathbf{a}_n$  is finitely compressible but, towards a contradiction,  $\mathbf{a}$  is not compressible. Then by Nadkarni's Theorem,  $\mathbf{a}$  admits an invariant measure and thus so does  $\mathbf{a}_n$ , contradicting the compressibility of  $\mathbf{a}_n$ .  $\square$

**Problem 3.5.7.** In [Theorem 3.5.3](#), can one replace (iii) by “ $\mathbf{a}$  is finitely compressible” and similarly for (iv).

**Remark 3.5.8.** It follows from [Theorem 3.5.3](#) that for a continuous action  $\mathbf{a}$  of a countable group on a compact Polish space, the property “ $\mathbf{a}$  has a finitely compressible (or finitely paradoxical) amplification” is a property of the induced equivalence relation  $E_{\mathbf{a}}$ . More precisely, if  $\mathbf{a}, \mathbf{b}$  are two continuous actions of groups  $\Gamma, \Delta$  on compact metrizable spaces  $X, Y$ , resp., and  $E_{\mathbf{a}} \cong_B E_{\mathbf{b}}$ , i.e.,  $E_{\mathbf{a}}, E_{\mathbf{b}}$  are Borel isomorphic, then  $\mathbf{a}$  admits a finitely compressible (or finitely paradoxical) amplification iff  $\mathbf{b}$  admits a finitely compressible (or finitely paradoxical) amplification. In view of [Problem 3.5.7](#), this may not be true for the property “ $\mathbf{a}$  is finitely compressible” or “ $\mathbf{a}$  is finitely paradoxical”. In fact one way to try to prove [Problem 3.5.7](#) is to search for two continuous actions  $\mathbf{a}, \mathbf{b}$  of countable groups  $\Gamma, \Delta$  on a compact metrizable space  $X$  with  $E_{\mathbf{a}} = E_{\mathbf{b}}$ , for which  $\mathbf{a}$  is finitely compressible (or finitely paradoxical) but  $\mathbf{b}$  is not.

**Remark 3.5.9.** [Theorem 3.5.3](#) fails if the space  $X$  is not compact. In fact there are even counterexamples with  $X$  Polish locally compact. Recall that an action of a group  $\Gamma$  on a set  $X$  is **amenable** if there is a finitely additive probability measure defined on all subsets of  $X$  and invariant under the action. Any action of a countable amenable group is amenable. Take now  $\Gamma$  to be a locally finite, infinite group and consider the left-translation action of  $\Gamma$  on itself. This action is not finitely compressible. Let then  $X = 2^{\mathbb{N}} \times \Gamma$  (with  $\Gamma$  discrete). This is Polish locally compact and  $\Gamma$  acts on it continuously by the action  $a$  given by  $\gamma \cdot (x, \delta) = (x, \gamma\delta)$ . This action is clearly compressible via the map  $T(x, \gamma) = (x, f(\gamma))$ , where  $f: \Gamma \rightarrow \Gamma$  is an injection with  $f(\Gamma) \neq \Gamma$ , so (i) in [Theorem 3.5.3](#) holds. On the other hand, all amplifications  $a_n$  are amenable, so not finitely paradoxical and (iv) in [Theorem 3.5.3](#) fails. Also all the actions  $a_n$  are not finitely compressible and (iii) in [Theorem 3.5.3](#) also fails.

In this counterexample the action  $\mathbf{a}$  is smooth. One can find another counterexample where the action  $\mathbf{a}$  is not smooth as follows: Let  $\Gamma$  be as before and consider again the translation action of  $\Gamma$  on itself. Let also  $\Gamma$  act on  $2^{\Gamma}$  by shift and consider the action  $\mathbf{b}$  of  $\Delta = \Gamma^2$  on  $X = 2^{\Gamma} \times \Gamma$  given by  $(\gamma, \delta) \cdot (x, \varepsilon) = (\gamma \cdot x, \delta\varepsilon)$ . This action is not smooth and is compressible but it is also amenable, since the action of each factor of  $\Delta$  is amenable on the corresponding space and therefore the action of  $\Delta$  is amenable by taking the product of finitely additive probability measures witnessing the amenability of these two actions. (By the product of a finitely additive probability measure  $\mu$  defined on all subsets of a set  $A$  and a finitely additive probability measure  $\nu$  defined on all subsets of a set  $B$ , we mean the finitely additive probability

measure  $\mu \times \nu$  on  $A \times B$  defined by  $\mu \times \nu(C) = \int_A \nu(C_x) d\mu(x)$ .) Also all the actions  $\mathbf{a}_n$  are not finitely compressible.

**Remark 3.5.10.** Using [Remark 3.5.1](#) it is easy to see that finite compressibility does not imply finite paradoxicality even for continuous actions of countable groups on compact Polish spaces. To see this, let  $G$  be a compact metrizable group containing a copy of  $\mathbb{Z}$  (e.g., the unit circle under multiplication) and consider the left-translation action of  $\mathbb{Z}$  on  $G$ .

**Remark 3.5.11.** Let  $E$  be a countable Borel equivalence relation on a standard Borel space  $X$ . We say that  $E$  is compressible (resp., finitely compressible, paradoxical, finitely paradoxical) if there is a Borel action  $\mathbf{a}$  of a countable group  $\Gamma$  on  $X$  with  $E = E_{\mathbf{a}}$  and  $\mathbf{a}$  is compressible (resp., finitely compressible, paradoxical, finitely paradoxical). Then it is easy to check that these conditions are equivalent. Indeed if  $E$  is compressible, then there is a smooth, aperiodic (i.e., having infinite classes) Borel equivalence relation  $F$  with  $F \subseteq E$ . Then  $F \cong_B \mathbb{R} \times I_{\mathbb{N}}$ , where  $H = \mathbb{R} \times I_{\mathbb{N}}$  is the equivalence relation on  $\mathbb{R} \times \mathbb{N}$  given by  $(x, m)H(y, n) \iff x = y$ . There is a transitive action of the free group with two generators  $\mathbb{F}_2$  on  $\mathbb{N}$  which is finitely paradoxical and thus a Borel action  $\mathbf{b}$  of  $\mathbb{F}_2$  on  $\mathbb{R} \times \mathbb{N}$  with  $F = E_{\mathbf{b}}$  which is finitely paradoxical. Fix also a Borel action  $\mathbf{c}$  of a countable group  $\Delta$  with  $E_{\mathbf{c}} = E$ . Then the action  $\mathbf{a}$  of  $\Gamma * \mathbb{F}_2$  that is equal to  $\mathbf{c}$  on  $\Gamma$  and  $\mathbf{b}$  on  $\mathbb{F}_2$  is finitely paradoxical and  $E_{\mathbf{a}} = E$ .

**Remark 3.5.12.** Ronnie Chen pointed out that (iv)  $\implies$  (ii) in [Theorem 3.5.3](#) can be also proved by using the cardinal algebra  $K(E \times I_{\mathbb{N}})$  as in [\[Che20\]](#) and the cancellation law for cardinal algebras.

Recall also that a CBER  $E$  admits an invariant measure iff *some* Borel action of a countable group that generates  $E$  has an invariant measure iff *every* Borel action of a countable group that generates  $E$  has an invariant measure (iff  $E$  is not compressible). On the other hand, there are aperiodic CBER  $E$  such that some Borel action of a countable group that generates  $E$  has an invariant finitely additive Borel probability measure but some other Borel action of a countable group that generates  $E$  has no invariant finitely additive Borel probability measure. For example, let  $E = E_t$ . There is a continuous action of  $\mathbb{F}_2$  on  $2^{\mathbb{N}}$  that generates  $E$  (see the proof of [Theorem 3.2.6](#)) and this action has no invariant finitely additive Borel probability measure by [Theorem 3.5.4](#). On the other hand,  $E_t$  is induced by a Borel action of  $\mathbb{Z}$  and this action has in fact an invariant finitely additive probability measure defined on all subsets of  $2^{\mathbb{N}}$ .

However in view of [Remark 3.5.11](#) we have the following equivalent formulation of existence of invariant measures for a CBER:

**Proposition 3.5.13.** *For every aperiodic CBER  $E$ ,  $E$  admits an invariant measure iff every Borel action of a countable group that generates  $E$  admits an invariant finitely additive Borel probability measure.*

(B) The preceding results in part (A) of this subsection can be generalized as follows.

Let  $\Gamma$  be a countable group and let  $\mathbf{a}$  be an action of  $\Gamma$  on a set  $X$ . Let also  $\mathcal{A}$  be an algebra of subsets of  $X$  invariant under this action. For  $A, B \in \mathcal{A}$ , let  $A \sim_{\mathcal{A}} B$  iff there are partitions  $A = \bigsqcup_{i=1}^n A_i, B = \bigsqcup_{i=1}^n B_i$ , where  $A_i, B_i \in \mathcal{A}$ , and  $\gamma_i \in \Gamma$  such that  $\gamma_i \cdot A_i = B_i$ . We say that the action is  **$\mathcal{A}$ -finitely compressible** if  $X \sim_{\mathcal{A}} Y$  with witnesses  $X_i, Y_i, \gamma_i$  as above, so that if  $T: X \rightarrow X$  is such that  $T(x) = \gamma_i \cdot x$ , for  $x \in X_i$ , then for every orbit  $C$  of the action,  $T(C) \subsetneq C$ . Also the action is  **$\mathcal{A}$ -finitely paradoxical** if there is a partition  $X = Y \sqcup Z$ , with  $Y, Z \in \mathcal{A}$  and  $X \sim_{\mathcal{A}} Y \sim_{\mathcal{A}} Z$ . The concept of an invariant finitely additive probability measure on  $\mathcal{A}$  is defined as before.

We extend the algebra  $\mathcal{A}$  to an algebra  $\mathcal{A}_n$  of subsets of  $X \times [n]$  by letting  $A \in \mathcal{A}_n \iff A = \bigcup_{i=1}^n A_i \times \{i\}$ , where  $A_i \in \mathcal{A}$ . We say that  $\mathbf{a}_n$  is  $\mathcal{A}$ -finitely compressible if it is  $\mathcal{A}_n$ -finitely compressible. Similarly we define what it means for  $\mathbf{a}_n$  to be  $\mathcal{A}$ -finitely paradoxical.

We now have the following generalization of [Theorem 3.5.3](#):

**Theorem 3.5.14.** *Let  $\mathbf{a}$  be a continuous action of a countable group  $\Gamma$  on a compact Polish space  $X$ . Let  $\mathcal{A}$  be an algebra of subsets of  $X$  which is invariant under the action and contains a basis for  $X$ . Then the following are equivalent:*

- (i) *there is no invariant finitely additive probability measure  $\mu$  on  $\mathcal{A}$ ;*
- (ii) *there is no invariant measure  $\nu$ ;*
- (iii) *an amplification of  $\mathbf{a}$  is  $\mathcal{A}$ -finitely compressible;*
- (iv) *an amplification of  $\mathbf{a}$  is  $\mathcal{A}$ -finitely paradoxical.*

The proof of [Theorem 3.5.14](#) is similar to the proof of [Theorem 3.5.3](#) using the following generalizations of [Theorem 3.5.4](#) and [Theorem 3.5.6](#).

**Theorem 3.5.15** ([\[Tse15, 5.3\]](#)). *Let  $\Gamma$  be a countable group and let  $\mathbf{a}$  be a continuous action of  $\Gamma$  on a second countable Hausdorff space  $X$ . Let  $\mathcal{A}$  be an algebra of subsets of  $X$  which is invariant under the action and contains a basis for  $X$  and a compact set  $K$ . If there is an invariant finitely additive probability measure  $\mu$  on  $\mathcal{A}$  with  $\mu(K) > 0$ , then there is an invariant (Borel probability, countably additive) measure  $\nu$ .*

**Theorem 3.5.16** ([TW16, 11.3]). *Let  $\Gamma$  be a countable group and let  $\mathbf{a}$  be an action of  $\Gamma$  on a set  $X$ . Let  $\mathcal{A}$  be an algebra of subsets of  $X$  invariant under this action. Then the following are equivalent:*

- (i) *there is no invariant finitely additive probability measure on  $\mathcal{A}$ ;*
- (ii) *there is a  $\mathcal{A}$ -finitely paradoxical amplification of  $\mathbf{a}$ .*

As a particular case of [Theorem 3.5.14](#) we have the following. Let  $\mathbf{a}$  be a continuous action of a countable group  $\Gamma$  on a zero-dimensional compact Polish space  $X$  (e.g., the Cantor space). Let  $\mathcal{C}$  be the algebra of clopen subsets of  $X$ . Then the following are equivalent:

- (i) there is no invariant finitely additive probability measure  $\mu$  on  $\mathcal{C}$ ;
- (ii) there is no invariant measure  $\nu$ ;
- (iii) an amplification of  $\mathbf{a}$  is  $\mathcal{C}$ -finitely compressible;
- (iv) an amplification of  $\mathbf{a}$  is  $\mathcal{C}$ -finitely paradoxical;
- (v)  $\mathbf{a}$  is compressible;
- (vi)  $\mathbf{a}$  is paradoxical;
- (vii) an amplification of  $\mathbf{a}$  is finitely compressible;
- (viii) an amplification of  $\mathbf{a}$  is finitely paradoxical.

Thus, rather surprisingly, for a continuous action of a countable group on a zero-dimensional compact Polish space, existence of a (countable Borel) paradoxical decomposition is equivalent to the existence of an amplification with a finite paradoxical decomposition using Borel pieces and also equivalent to the existence of an amplification with a finite paradoxical decomposition using *clopen* pieces.

### 3.6 Realizations by subshifts

*In this and the next two sections, unless it is otherwise stated or clear from the context, we assume all groups to be countable.*

If  $\Gamma$  is a group a  $\Gamma$ -**flow** is a continuous action of  $\Gamma$  on a compact Polish space. A **subflow** of a  $\Gamma$ -flow is the restriction of the action to a nonempty closed invariant subset.

Recall that for a group  $\Gamma$ , a set  $L$  equipped with a  $\Gamma$ -action, and a set  $X$ , the canonical **shift action** of  $\Gamma$  on  $X^L$  is given by

$$(\gamma \cdot x)_l = x_{\gamma^{-1}l}.$$

When  $X$  is a topological space, the restriction of the shift action to a nonempty closed invariant set  $F \subseteq X^\Gamma$  is called a **subshift**. We often refer to  $F$  itself as a subshift.

If  $X$  is compact Polish, this is a  $\Gamma$ -flow and we denote it by  $\mathbf{s}_{L,X}$ . In particular,  $\mathbf{s}_{\Gamma,X}$  is the shift action of  $\Gamma$ , where  $\Gamma$  acts on itself by left multiplication.

For  $\Gamma$ -flows  $\mathbf{a}$  and  $\mathbf{b}$  on  $X$  and  $Y$  respectively, a  $\Gamma$ -map  $\mathbf{a} \rightarrow \mathbf{b}$  is a  $\Gamma$ -equivariant continuous function  $X \rightarrow Y$ . Let  $\text{Hom}_{\Gamma}(\mathbf{a}, \mathbf{b})$  denote the set of  $\Gamma$ -maps  $\mathbf{a} \rightarrow \mathbf{b}$ .

Below for any action  $\mathbf{a}$ , we denote by  $E_{\mathbf{a}}$  the induced orbit equivalence relation.

### (A) Coinduction and generators

Let  $\Gamma \leq \Delta$  be groups. Given a  $\Delta$ -flow  $\mathbf{b}$ , we denote the  $\Gamma$ -restriction of  $\mathbf{b}$  by  $\mathbf{b}|_{\Gamma}$ .

Given a  $\Gamma$ -flow  $\mathbf{a}$  on  $X$ , the **coinduced  $\Delta$ -flow** of  $\mathbf{a}$ , denoted by  $\text{CInd}_{\Gamma}^{\Delta}(\mathbf{a})$ , is the  $\Delta$ -subflow of  $\mathbf{s}_{\Delta,X}$  on the subspace

$$\{x \in X^{\Delta} : \forall \gamma \in \Gamma \forall \delta \in \Delta [x_{\delta\gamma} = \gamma^{-1} \cdot x_{\delta}]\}.$$

In particular,  $\mathbf{s}_{\Gamma,X}$  is isomorphic to  $\text{CInd}_{1}^{\Gamma}(\mathbf{s}_{1,X})$ , where  $1$  is the trivial group (note that  $\mathbf{s}_{1,X}$  is the 1-flow on  $X$ ).

There is a natural bijection

$$\text{Hom}_{\Delta}(\mathbf{b}, \text{CInd}_{\Gamma}^{\Delta}(\mathbf{a})) \cong \text{Hom}_{\Gamma}(\mathbf{b}|_{\Gamma}, \mathbf{a})$$

taking  $f$  to the map  $y \mapsto (f(y))_1$ .

Let  $\mathbf{a}$  and  $\mathbf{b}$  be flows for  $\Gamma$  and  $\Delta$  respectively. A  $\Gamma$ -map  $f : \mathbf{b}|_{\Gamma} \rightarrow \mathbf{a}$  is an  **$\mathbf{a}$ -generator** for  $\mathbf{b}$  if its corresponding  $\Delta$ -map  $\mathbf{b} \rightarrow \text{CInd}_{\Gamma}^{\Delta}(\mathbf{a})$  is injective. Explicitly,  $f$  is an  **$\mathbf{a}$ -generator** for  $\mathbf{b}$  if for every  $x, x' \in X_{\mathbf{a}}$ , if  $f(\delta \cdot x) = f(\delta \cdot x')$  for every  $\delta$ , then  $x = x'$ .

We note the following facts:

(1) Let  $\mathbf{a}$  be a  $\Gamma$ -flow on  $X$ , and let  $n \geq 2$ . Considering  $n$  as a discrete space, an  $\mathbf{s}_{1,n}$ -generator for  $\mathbf{a}$  coincides with the usual notion of a **clopen  $n$ -generator** for  $\mathbf{a}$ , that is, a partition  $(A_i)_{i < n}$  of  $X$  into clopen sets such that for every  $x, x' \in X$ , if for every  $\gamma \in \Gamma$  and every  $i < n$  we have

$$\gamma \cdot x \in A_i \iff \gamma \cdot x' \in A_i,$$

then  $x = x'$ . Equivalently  $\mathbf{a}$  admits a clopen  $n$ -generator iff it is (topologically) isomorphic to a subshift of  $n^{\Gamma}$ .

(2) Every injective  $\Gamma$ -map  $\mathbf{b}|_{\Gamma} \hookrightarrow \mathbf{a}$  is an  **$\mathbf{a}$ -generator** for  $\mathbf{b}$ .

(3) If  $\mathbf{b} = \text{CInd}_{\Gamma}^{\Delta}(\mathbf{a})$ , then the map  $y \mapsto y_1$  is an  **$\mathbf{a}$ -generator** for  $\mathbf{b}$ , since it corresponds to the identity on  $\text{CInd}_{\Gamma}^{\Delta}(\mathbf{a})$ .

(4) Let  $\Gamma \leq \Delta \leq \Lambda$  be groups with flows  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  respectively. If  $\mathbf{c}$  has a  **$\mathbf{b}$ -generator**  $f$ , and  $\mathbf{b}$  has an  **$\mathbf{a}$ -generator**  $g$ , then the composition  $f \circ g$  is an  **$\mathbf{a}$ -generator**

for  $\mathbf{c}$ . To see this, let  $x, x' \in X_{\mathbf{a}}$  and suppose that  $f(g(\lambda \cdot x)) = f(g(\lambda \cdot x'))$  for every  $\lambda$ . Then for every  $\delta$  and every  $\lambda$ , we have  $f(\delta \cdot g(\lambda \cdot x)) = f(\delta \cdot g(\lambda \cdot x'))$ . Thus since  $f$  is a generator, we have  $g(\lambda \cdot x) = g(\lambda \cdot x')$  for every  $\lambda$ . Since  $g$  is a generator, we have  $x = x'$ .

Below we call a flow **compressible** iff the induced equivalence relation is compressible. Equivalently by Nadkarni's Theorem (see, e.g., [Kec21b, 5.3]) this means that the flow admits no invariant Borel probability measure.

**Proposition 3.6.1.** *Let  $\Gamma \leq \Delta$  be groups, let  $\mathbf{a}$  be a  $\Gamma$ -flow, and let  $\mathbf{b}$  be a  $\Delta$ -flow.*

(i) *Suppose there is a  $\Gamma$ -map  $\mathbf{b}|_{\Gamma} \rightarrow \mathbf{a}$ . If  $\mathbf{a}$  is compressible, then  $\mathbf{b}$  is compressible.*

(ii) *Suppose  $\mathbf{b}$  has an  $\mathbf{a}$ -generator. If  $\mathbf{a}$  has a clopen  $n$ -generator, then  $\mathbf{b}$  has a clopen  $n$ -generator.*

(iii) *Suppose  $\mathbf{b}|_{\Gamma} = \mathbf{a}$ . If  $\mathbf{a}$  is minimal, then  $\mathbf{b}$  is minimal.*

*Proof.* (i) If  $\mu$  is an invariant Borel probability measure for  $\mathbf{b}$ , then it is invariant for  $\mathbf{b}|_{\Gamma}$ , so its pushforward to  $\mathbf{a}$  is invariant.

(ii) If  $\mathbf{a}$  has a clopen  $n$ -generator, then it has an  $\mathbf{s}_{1,n}$ -generator. Composing them gives a clopen  $n$ -generator for  $\mathbf{b}$ .

(iii) This is obvious. □

**Corollary 3.6.2.** *Let  $\Gamma \leq \Delta$  be groups. The following properties of a  $\Gamma$ -flow pass to its coinduced  $\Delta$ -flow:*

(i) *Compressibility.*

(ii) *Existence of a clopen  $n$ -generator.*

*Proof.* Take  $\mathbf{b} = \text{CInd}_{\Gamma}^{\Delta}(\mathbf{a})$  in Proposition 3.6.1. □

## (B) Jumps

Let  $\Gamma$  and  $\Lambda$  be groups, and let  $L$  be a countable  $\Lambda$ -set. The **unrestricted wreath product** is the group  $\Gamma \wr_L \Lambda$  defined by

$$\Gamma \wr_L \Lambda := \Gamma^L \rtimes \Lambda = \langle \Gamma^L, \Lambda : \lambda \gamma \lambda^{-1} = \lambda \cdot \gamma \rangle$$

If  $L = \Lambda$  with the left-translation action, then we omit the subscript and write  $\Gamma \wr \Lambda$ . Denote by  $\Gamma^{\oplus L}$  the subgroup of  $\Gamma^L$  consisting of those elements which are the identity on cofinitely many coordinates. Note that the shift  $\Lambda$ -action on  $\Gamma^L$  preserves  $\Gamma^{\oplus L}$ . The **restricted wreath product** is the subgroup  $\Gamma \wr_L^{\oplus} \Lambda$  of  $\Gamma \wr_L \Lambda$  generated by  $\Gamma^{\oplus L}$  and  $\Lambda$ . If  $L$  is a transitive  $\Lambda$ -set, and  $S$  and  $T$  are generating sets for  $\Gamma$  and  $\Lambda$  respectively, then  $S \cup T$  generates  $\Gamma \wr_L^{\oplus} \Lambda$  (see [HR94, 2.3]).

Let  $E$  be a CBER on  $X$ . The **unrestricted  $L$ -jump** of  $E$ , denoted  $E^{[L]}$  is the Borel equivalence relation on  $X^L$  defined by

$$x E^{[L]} y \iff \exists \lambda [\lambda \cdot x E^L y].$$

(see [CC20] for more uses of this jump). Let  $E^{\oplus L}$  be the subequivalence relation of the product equivalence relation  $E^L$  consisting of pairs which are equal on cofinitely many coordinates. The **restricted  $L$ -jump** of  $E$ , is the subequivalence relation  $E^{\oplus[L]}$  of  $E^{[L]}$  which is the intersection of  $E^{[L]}$  and  $E^{\oplus L}$ .

Given a Borel embedding  $E \sqsubseteq_B F$  via a map  $X \rightarrow Y$ , the induced map  $X^L \rightarrow Y^L$  witnesses the Borel embeddings  $E^{[L]} \sqsubseteq_B F^{[L]}$  and  $E^{\oplus[L]} \sqsubseteq_B F^{\oplus[L]}$ .

Let  $\mathbf{a}$  be a  $\Gamma$ -flow on  $X$ . Let  $\mathbf{a}^L$  be the  $\Gamma^L$ -flow on  $X^\Lambda$  defined by  $(\gamma \cdot \mathbf{x})_l = \gamma_l \cdot x_l$ . We have  $E_{\mathbf{a}^L} = (E_{\mathbf{a}})^L$ . The **unrestricted  $L$ -jump** of  $\mathbf{a}$ , denoted  $\mathbf{a}^{[L]}$ , is the unique  $\Gamma \wr_L \Lambda$ -flow on  $X^L$  which simultaneously extends both the  $\Gamma^L$ -flow  $\mathbf{a}^L$  and the  $\Lambda$ -flow  $\mathbf{s}_{L,X}$  on  $X^L$ . Explicitly, the action is given by  $(\gamma \lambda \cdot \mathbf{x})_l = \gamma_{\lambda'} x_{\lambda^{-1}l}$ . We have  $E_{\mathbf{a}^{[L]}} = (E_{\mathbf{a}})^{[L]}$ , since

$$x E_{\mathbf{a}^{[L]}} y \iff \exists \gamma \exists \lambda [\gamma \lambda \cdot x = y] \iff \exists \lambda [\lambda \cdot x E_{\mathbf{a}^L} y] \iff \exists \lambda [\lambda \cdot x E_{\mathbf{a}}^L y]$$

Let  $\mathbf{a}^{\oplus L}$  be the  $\Gamma^{\oplus L}$ -flow  $\mathbf{a}^L \upharpoonright_{\Gamma^{\oplus L}}$ . We have  $E_{\mathbf{a}^{\oplus L}} = E_{\mathbf{a}}^{\oplus L}$ . The **restricted  $L$ -jump** of  $\mathbf{a}$ , denoted  $\mathbf{a}^{\oplus[L]}$ , is the restriction  $\mathbf{a}^{[L]} \upharpoonright_{\Gamma^{\oplus L}}$ . We have  $E_{\mathbf{a}^{\oplus[L]}} = (E_{\mathbf{a}})^{\oplus[L]}$ .

If  $L$  is a transitive  $\Lambda$ -set, then for any  $l_0 \in L$ , the map  $\mathbf{x} \mapsto x_{l_0}$  is an  $\mathbf{a}$ -generator for  $\mathbf{a}^{\oplus[L]}$ , since if  $(\lambda \cdot \mathbf{x})_{l_0} = (\lambda \cdot \mathbf{y})_{l_0}$  for every  $\lambda$ , then by transitivity, we have  $x_l = y_l$  for every  $l \in L$ , and thus  $x = y$ .

**Proposition 3.6.3.** *The following properties of a  $\Gamma$ -flow pass to its restricted  $L$ -jump:*

- (i) *Compressibility.*
- (ii) *Existence of a clopen  $n$ -generator.*
- (iii) *Minimality.*

*Proof.* Let  $\mathbf{a}$  be a  $\Gamma$ -flow.

Since  $\mathbf{a}^{\oplus[L]}$  has an  $\mathbf{a}$ -generator, the first two properties follow from [Proposition 3.6.1](#).

If  $\mathbf{a}$  is minimal  $\Gamma$ -flow, then  $\mathbf{a}^{\oplus L}$  is a minimal  $\Gamma^{\oplus L}$ -flow. Since  $\mathbf{a}^{\oplus[L]} \upharpoonright_{\Gamma^{\oplus L}} = \mathbf{a}^{\oplus L}$ , we have by [Proposition 3.6.1](#) that  $\mathbf{a}^{\oplus[L]}$  is minimal.  $\square$

### (C) Realizations by minimal subshifts

A flow is **orbit-universal** if its orbit equivalence relation is a universal CBER.

Let  $E(L, \mathbb{R})$  denote the orbit equivalence relation of the shift action  $\Lambda \curvearrowright \mathbb{R}^L$ .



**Theorem 3.6.4.** *Let  $\Gamma$  and  $\Lambda$  be countable groups. Let  $L$  be a countable  $\Lambda$ -set, and let  $\mathbf{a}$  be a  $\Gamma$ -flow on  $X$ , with  $X$  uncountable. Then there is a Borel injection  $f : \mathbb{R}^L \rightarrow X^L$  which simultaneously witnesses  $E(L, \mathbb{R}) \sqsubseteq_B E_{\mathbf{a}^{\oplus[L]}}$  and  $E(L, \mathbb{R}) \sqsubseteq_B E_{\mathbf{a}^{[L]}}$ . In particular, for every group  $G$  (no definability condition required) with  $\Gamma \wr_L^\oplus \Lambda \leq G \leq \Gamma \wr_L \Lambda$ , the map  $f$  witnesses  $E(L, \mathbb{R}) \sqsubseteq_B E_{\mathbf{a}^{[L]} \upharpoonright_G}$ .*

*In particular, if  $E(L, \mathbb{R})$  is universal and  $\Delta$  is a countable group with  $\Gamma \wr_L^\oplus \Lambda \leq \Delta \leq \Gamma \wr_L \Lambda$ , then  $\mathbf{a}^{[L]} \upharpoonright_\Delta$  is orbit-universal.*

*Proof.* Since  $X$  is uncountable, there is a Borel map  $\mathbb{R} \rightarrow X$  witnessing  $\text{id}_{\mathbb{R}} \sqsubseteq_B E_{\mathbf{a}}$ . Let  $f : \mathbb{R}^L \rightarrow X^L$  be the induced map. Then  $f$  witnesses

$$E(L, \mathbb{R}) = (\text{id}_{\mathbb{R}})^{\oplus[L]} \sqsubseteq_B (E_{\mathbf{a}})^{\oplus[L]} = E_{\mathbf{a}^{\oplus[L]}}$$

and also

$$E(L, \mathbb{R}) = (\text{id}_{\mathbb{R}})^{[L]} \sqsubseteq_B (E_{\mathbf{a}})^{[L]} = E_{\mathbf{a}^{[L]}}$$

□

**Corollary 3.6.5.** *Let  $\Gamma$  and  $\Lambda$  be groups, and suppose that there is a countable transitive  $\Lambda$ -set  $L$  such that  $E(L, \mathbb{R})$  is a universal CBER. Let  $\Delta$  be a countable group with a factor  $\Delta'$  such that  $\Gamma \wr_L^\oplus \Lambda \leq \Delta' \leq \Gamma \wr_L \Lambda$ . Then there is an orbit-universal minimal  $\Delta'$ -flow with a clopen 2-generator. If  $\Gamma$  is non-amenable, then this flow can be taken to be compressible.*

*Proof.* It suffices to consider the case where  $\Delta' = \Delta$ . Let  $\mathbf{a}$  be an uncountable minimal  $\Gamma$ -flow with a 2-generator; for instance, take a minimal subshift of a free subshift of  $2^\Gamma$  (these exist by [GJS09]; see also [Ber17]). If  $\Gamma$  is non-amenable, by Theorem 3.7.1 below we can take  $\mathbf{a}$  to be compressible and then pass to a subflow to ensure minimality.

Now consider the  $\Gamma \wr_L^\oplus \Lambda$ -flow  $\mathbf{a}^{\oplus[\Lambda]}$ . By Proposition 3.6.3, this is minimal and has a 2-generator, and is compressible if  $\Gamma$  is non-amenable. Orbit-universality follows from Theorem 3.6.4. □

**Corollary 3.6.6.** (i) *There is an orbit-universal minimal subshift of  $2^{\mathbb{F}_3}$ .*

(ii) *There is a compressible orbit-universal minimal subshift of  $2^{\mathbb{F}_4}$ .*

*In particular any compressible, universal CBER admits a minimal, compact action realization which is in fact a minimal subshift of  $2^{\mathbb{F}_4}$ .*

*Proof.* Recall that  $E(\mathbb{F}_2, 2)$  is a universal CBER.

(i)  $\mathbb{F}_3$  has the factor  $\mathbb{Z} \wr^\oplus \mathbb{F}_2$ , so by Corollary 3.6.5,  $\mathbb{Z} \wr^\oplus \mathbb{F}_2$  admits an orbit-universal minimal flow with a clopen 2-generator.

(ii)  $\mathbb{F}_4$  has the factor  $\mathbb{F}_2 \wr^\oplus \mathbb{F}_2$ , so since  $\mathbb{F}_2$  is non-amenable, by Corollary 3.6.5,  $\mathbb{F}_2 \wr^\oplus \mathbb{F}_2$  admits a compressible orbit-universal minimal flow with a clopen 2-generator, □

**Problem 3.6.7.** Does [Corollary 3.6.6](#) hold with  $\mathbb{F}_2$  instead of  $\mathbb{F}_3, \mathbb{F}_4$ ?

By [Corollary 3.6.5](#), it suffices to find some  $\Gamma$  and  $\Lambda \geq \mathbb{F}_2$  such that there is a 2-generated group between  $\Gamma \wr_L^\oplus \Lambda$  and  $\Gamma \wr_L \Lambda$  (and  $\Gamma$  non-amenable if we want compressibility).

In view of [Theorem 3.2.6](#) and [Corollary 3.6.6](#) one can ask whether the following very strong realization result is true:

**Problem 3.6.8.** Does every non-smooth aperiodic CBER have a realization as a subshift of  $2^\Gamma$  for some group  $\Gamma$ ? Also does it have a realization as a minimal subshift?

### (D) Minimal subshift universality

Let  $\Gamma$  be a countable group. We say that  $\Gamma$  is **minimal subshift universal** if there is a minimal subshift  $K$  of  $2^\Gamma$  such that if  $E$  is the shift equivalence relation on  $2^\Gamma$ , then  $E|K$  is universal. We note the following equivalent formulation of this notion. Recall that a point  $x \in 2^\Gamma$  is minimal if  $\overline{\Gamma \cdot x}$  is a minimal  $\Gamma$ -flow; equivalently, for every finite  $A \subseteq \Gamma$ , the set  $\{\gamma \in \Gamma : (\gamma \cdot x)|_A = x|_A\}$  is **left syndetic**, i.e. finitely many left translates of it cover  $\Gamma$  (see [\[dV93, IV\(1.2\)\]](#)).

**Proposition 3.6.9.** Let  $\Gamma$  be a countable group. Then the following are equivalent:

- (i)  $\Gamma$  is minimal subshift universal;
- (ii) There is a minimal  $\Gamma$ -flow which admits a clopen 2-generator and such that the induced equivalence relation is universal;
- (iii) If  $M$  is the set of minimal points in  $2^\Gamma$  and  $E$  is the shift equivalence relation, then  $E|M$  is universal

*Proof.* Clearly (i) implies (ii) implies (iii). Assume now (iii). Consider the Borel map  $f$  that sends  $x \in M$  to the closure of its orbit (which is an element of the space of compact subsets of  $2^\Gamma$ ). Then by [\[MSS16, Theorem 3.1\]](#) there is some  $K$  such that  $E|f^{-1}(K)$  is universal. But clearly  $f^{-1}(K) = K$ , so  $K$  is a minimal subshift, thus (i) holds.  $\square$

Clearly if  $\Gamma$  is minimal subshift universal and there is a surjective homomorphism of  $\Delta$  on  $\Gamma$ , then  $\Delta$  is minimal subshift universal. The existence of a minimal subshift universal group was first proved by Brandon Seward, who showed that  $\mathbb{F}_\infty$  has this property. [Corollary 3.6.5](#) shows that any wreath product  $\Gamma \wr \Lambda$ , where  $\Gamma$  is infinite and  $\Lambda$  contains  $\mathbb{F}_2$ , is minimal subshift universal and in particular by [Corollary 3.6.6](#),  $\mathbb{F}_3$  is minimal subshift universal. We include below Seward's proof for  $\mathbb{F}_\infty$  (and somewhat more), with his permission, as it is based on a very different method.

**Theorem 3.6.10** (Seward). *Let  $\Gamma$  be a group. Then  $E(\Gamma, 2) \sqsubseteq_B E(\Gamma * \mathbb{F}_\infty, 2) \upharpoonright_M$ , where  $M \subseteq 2^{\Gamma * \mathbb{F}_\infty}$  denotes the set of minimal points.*

*Proof.* We start with the following lemma.

**Lemma 3.6.11.** *Let  $\Gamma$  be a group and let  $A \subseteq \Gamma$  be a finite subset. There is a  $\Gamma$ -equivariant Borel embedding  $x \mapsto x'$  from  $E(\Gamma, 2)$  to  $E(\Gamma * \mathbb{Z}, 2)$  such that for every  $x \in 2^\Gamma$ ,*

$$(i) \ x' \upharpoonright_\Gamma = x \text{ (i.e. } x' \text{ extends } x),$$

$$(ii) \ ((\Gamma * \mathbb{Z}) \cdot x') \upharpoonright_\Gamma \subseteq \Gamma \cdot x,$$

(iii) *the  $\mathbb{Z}$ -action on  $(\Gamma * \mathbb{Z}) \cdot x'$  factors via the restriction map  $2^{\Gamma * \mathbb{Z}} \rightarrow 2^B$  to a transitive action (on the image).*

*Proof.* Fix an enumeration of  $\Gamma$ , and let  $t$  denote the generator of  $\mathbb{Z}$ . For every nonempty subset  $P$  of  $2^A$ , fix a transitive permutation  $\sigma_P$  of  $P$ . Let  $x \in 2^\Gamma$ , and let  $P_x = (\Gamma \cdot x) \upharpoonright_A$  be the set of  $A$ -patterns appearing in  $x$ . We define  $x'$  inductively on left cosets of  $\Gamma$ , starting with  $x' \upharpoonright_\Gamma = x$ .

Let  $tw$  be a reduced word in  $2^{\Gamma * \mathbb{Z}}$  for which  $(w \cdot x') \upharpoonright_\Gamma$  is already defined. Then set  $(tw \cdot x') \upharpoonright_\Gamma = \gamma \cdot ((w \cdot x') \upharpoonright_\Gamma)$ , where  $\gamma$  is minimal with  $(\gamma \cdot ((w \cdot x') \upharpoonright_\Gamma)) \upharpoonright_A = \sigma_{P_x}((w \cdot x') \upharpoonright_A)$ .

Similarly, if  $t^{-1}w$  is a reduced word for which  $(w \cdot x') \upharpoonright_\Gamma$  is defined, then set  $(t^{-1}w \cdot x') \upharpoonright_\Gamma = \gamma^{-1} \cdot ((w \cdot x') \upharpoonright_\Gamma)$ , where  $\gamma$  is minimal with  $(\gamma^{-1} \cdot ((w \cdot x') \upharpoonright_\Gamma)) \upharpoonright_A = \sigma_{P_x}^{-1}((w \cdot x') \upharpoonright_A)$ .  $\square$

Let now  $t_0, t_1, t_2, \dots$  be the free generators of  $\mathbb{F}_\infty$ , and let  $\Gamma_n = \Gamma * \langle t_i \rangle_{i < n}$  (this includes the case  $n = \infty$ ). Let  $(A_n)_n$  be an exhaustive increasing sequence of finite subsets of  $\Gamma_\infty$  such that  $A_n \subseteq \Gamma_n$ . For every  $n$ , apply the lemma with  $\Gamma_n$  and  $A_n$  to obtain a Borel embedding  $E(\Gamma_n, 2) \sqsubseteq_B E(\Gamma_{n+1}, 2)$ . Given  $x_0 \in 2^{\Gamma_0}$ , let  $x_1$  denote the extension to  $2^{\Gamma_1}$  of  $x_0$ , let  $x_2$  denote the extension to  $2^{\Gamma_2}$  of  $x_1$ , and so on for  $x_n \in 2^{\Gamma_n}$ . Let  $x_\infty = \bigcup_n x_n$ . We claim that for every  $n$  and every  $m > n$  (including  $m = \infty$ ), the  $\langle t_n \rangle$ -action on  $\Gamma_m \cdot x_m$  factors via the restriction map  $2^{\Gamma_m} \rightarrow 2^{A_n}$  to a transitive  $\langle t_n \rangle$ -action on the image. It suffices to show this for finite  $m$ . We proceed by induction on  $m$ , for which the base case  $m = n + 1$  holds by the lemma. Now suppose that this holds for  $m$ . Let  $\gamma \in \Gamma_{m+1}$ . Then by the lemma,  $(\gamma \cdot x_{m+1}) \upharpoonright_{\Gamma_m} = h \cdot x_m$  for some  $h \in \Gamma_m$ , and thus

$$(t_m \gamma \cdot x_{m+1}) \upharpoonright_{\Gamma_m} = t_m \cdot ((\gamma \cdot x_{m+1}) \upharpoonright_{\Gamma_m}) = t_m \cdot (h \cdot x_m) = t_m h \cdot x_m$$

so

$$(t_m \gamma \cdot x_{m+1}) \upharpoonright_{A_n} = (t_m \gamma \cdot x_{m+1} \upharpoonright_{\Gamma_m}) \upharpoonright_{A_n} = (t_m h \cdot x_m) \upharpoonright_{A_n}$$

which only depends on  $(h \cdot x_m) \upharpoonright_{A_n} = (\gamma \cdot x_{m+1}) \upharpoonright_{A_n}$ , so the  $\langle t_n \rangle$ -action factors through  $2^{\Gamma_{m+1}} \rightarrow 2^{A_n}$ , and the action is clearly still transitive.

We show that the map  $x_0 \mapsto x_\infty$  is the desired embedding. It is clearly a  $\Gamma_0$ -invariant Borel injection. To see that it is a cohomomorphism, if  $(x_\infty, y_\infty) \in E(\Gamma_\infty, 2)$ , then  $x_\infty = \gamma \cdot y_\infty$  for some  $\gamma \in \Gamma_\infty$ . Now  $\gamma \in \Gamma_n$  for some  $n$ , so  $x_n = \gamma \cdot y_n$ , and thus  $(x_n, y_n) \in E(\Gamma_n, 2)$ . Since each extension map is a cohomomorphism, we have  $(x_0, y_0) \in E(\Gamma_0, 2)$ .

It remains to show that the image lies in  $M$ . Fix  $x_\infty$  and let  $A \subseteq \Gamma_\infty$ . We show that the set  $\{\gamma \in \Gamma_\infty : (\gamma \cdot x_\infty)|_A = x_\infty|_A\}$  is left syndetic. By enlarging  $A$ , we can assume that  $A = A_n$  for some  $n$ . Let  $T = \{(t_n)^k : 0 \leq k < 2^{|A_n|}\}$ . Now let  $\gamma \in \Gamma_\infty$ . Then by transitivity, there is some  $0 \leq k < 2^{|A_n|}$  for which  $((t_n)^k \gamma \cdot x)|_{A_n} = x_\infty|_{A_n}$ , so we are done.  $\square$

We can now restate [Problem 3.6.7](#), in a more general form, as follows:

**Problem 3.6.12.** *Is  $\mathbb{F}_2$  minimal subshift universal? More generally, is every group that contains  $\mathbb{F}_2$  minimal subshift universal?*

### 3.7 Subshifts as tests for amenability

It is well known that a group  $\Gamma$  is amenable iff every  $\Gamma$ -flow admits an invariant Borel probability measure. Call a class  $\mathcal{F}$  of  $\Gamma$ -flows a **test for amenability** for  $\Gamma$  if  $\Gamma$  is amenable provided that every  $\Gamma$ -flow in  $\mathcal{F}$  admits an invariant Borel probability measure. In [\[GdlH97\]](#) a compact metrizable space  $X$  is called a **test space** for the amenability of  $\Gamma$  if the class of all  $\Gamma$ -flows on  $X$  is a test for amenability for  $\Gamma$ . Giordano and de la Harpe show in [\[GdlH97\]](#) that the Cantor space  $2^{\mathbb{N}}$  is a test space for amenability of any group. Equivalently this says that the class of all subshifts of  $(2^{\mathbb{N}})^\Gamma$  is a test of amenability for  $\Gamma$ . We show next that the strongest result along these lines is actually true, namely that the class of all subshifts of  $2^\Gamma$  is a test of amenability for  $\Gamma$ . This gives another characterization of amenability.

**Theorem 3.7.1.** *Let  $\Gamma$  be a group. Then  $\Gamma$  is amenable iff every subshift of  $2^\Gamma$  admits an invariant Borel probability measure.*

*Proof.* We have to show that if  $\Gamma$  is not amenable then there is a compressible subshift of  $2^\Gamma$ . We will first give a proof for the case that  $\Gamma$  contains  $\mathbb{F}_2$ , which is much simpler, and then give the full proof for arbitrary non-amenable  $\Gamma$ .

*Proof when  $\Gamma \geq \mathbb{F}_2$ .*

It suffices to find a compressible  $\mathbb{F}_2$ -flow with a 2-generator, since if  $\mathbf{a}$  is such an  $\mathbb{F}_2$ -flow, then  $\text{CInd}_{\mathbb{F}_2}^\Gamma(\mathbf{a})$  is a compressible  $\Gamma$ -flow with a 2-generator by [Corollary 3.6.2](#). For the existence of such a  $\mathbb{F}_2$ -flow, see the proof of [Theorem 3.2.6](#).

*Proof for all non-amenable  $\Gamma$ .*

By nonamenability, there is a finite symmetric subset  $S \subseteq \Gamma$  containing 1 such that:

- (i) for every finite  $F \subseteq \Gamma$ , we have  $|FS| \geq 2|F|$ ;
- (ii) there is an integer  $n$  with

$$4 + 3 \log_2(|S|) \leq n \leq \frac{|S| - 6}{3 \log_2(|S|)}$$

- (iii) there is some  $r \in S$  with  $r^2 \neq 1$ .

Let  $T = S^n$ . Given a point  $x \in (T \sqcup \{*\})^\Gamma$ , let  $\text{Supp}(x)$  denote the set of  $\gamma \in \Gamma$  such that  $x_\gamma \neq *$ . Let  $X$  be the subshift of  $(T \sqcup \{*\})^\Gamma$  such that  $x \in X$  iff the following hold:

- (i)  $\text{Supp}(x)$  is maximal right  $S^3$ -disjoint (a subset  $A \subseteq \Gamma$  is **right  $S^3$ -disjoint** if for any distinct  $a, a' \in \text{Supp}(x)$ , we have  $a' \notin aS^3$ );
- (ii) the function  $\gamma \mapsto \gamma x_\gamma$  is a 2-to-1 surjection from  $\text{Supp}(x)$  onto  $\text{Supp}(x)$ .

We claim that  $X$  is the desired subshift. We first recall a fact from graph theory.

**Lemma 3.7.2.** *Let  $G$  be a locally finite (not necessarily simple) graph with vertex set  $V$ , such that every finite  $F \subseteq V$  satisfies  $|N_G(F)| \geq k|F|$ , where  $N_G(F)$  denotes the set of neighbours of  $F$ . Then there is a  $k$ -to-1 surjection  $p: V \rightarrow V$  such that for every  $v \in V$ , there is an edge from  $v$  to  $p(v)$ .*

*Proof.* Consider the bipartite graph  $B$  with bipartition  $(V_l, V_r)$ , where  $V_l = V_r = V$ , and where there is an edge from  $v \in V_l$  to  $w \in V_r$  if  $vw$  is an edge in  $V$ . Then every finite  $F \subseteq V_l$  satisfies  $|N_B(F)| \geq |F|$ , and every finite  $F \subseteq V_r$  satisfies  $|N_B(F)| \geq k|F|$ , so by Hall's theorem [TW16, C.4(b)], there are matchings  $(M_i)_{i < k}$  such that every vertex in  $V_l$  is covered by a unique  $M_i$ , and every vertex in  $V_r$  is covered by every  $M_i$ . Then  $\bigcup_{i < k} M_i$  is (the graph of) the desired  $k$ -to-1 surjection.  $\square$

We show that  $X$  is nonempty. Let  $A \subseteq \Gamma$  be any maximal right  $S^3$ -disjoint subset, and consider  $A$  as a (non-simple) graph where  $a$  and  $a'$  are adjacent iff  $a' \in aT$ . Let  $F \subseteq A$  be a finite subset. By maximality of  $A$ , every element of  $FS^{n-3}$  is within  $S^3$  of some element of  $FT \cap A$ , and thus

$$|FT \cap A| \geq \frac{|FS^{n-3}|}{|S^3|} \geq \frac{2^{n-3}|F|}{|S|^3} \geq 2|F|$$

by our choice of  $n$ . Thus by [Lemma 3.7.2](#), there is a 2-to-1 surjection  $p: A \rightarrow A$  such that  $p(a) \in aT$  for every  $a \in A$ . Define  $x \in X^\Gamma$  by

$$x_\gamma = \begin{cases} \gamma^{-1}p(\gamma) & \gamma \in A, \\ * & \text{otherwise.} \end{cases}$$

Then  $x \in X$ .

Next, we show that  $X$  is a compressible subshift. Let  $Y \subseteq X$  be the set of  $x \in X$  with  $1 \in \text{Supp}(x)$ . Consider the Borel map  $Y \rightarrow Y$  defined by  $y \mapsto y_1^{-1} \cdot y$ . This is a 2-to-1 surjection, since the preimage of  $y \in Y$  is the set  $\{\gamma^{-1} \cdot y : \gamma y_\gamma = 1\}$ . Thus  $Y$  is a compressible subset, so since  $Y$  is a complete section,  $X$  is also a compressible subset.

It remains to show that there is a clopen 2-generator.

Recall that if  $G$  is a finite graph with maximum degree  $d$ , then every independent set  $I \subseteq G$  can be extended to an independent set of size at least  $\frac{|G|}{d+1}$ . To see this, let  $I = I_0 \subseteq I_1 \subseteq \dots \subseteq I_k$  be a maximal chain of independent sets in  $G$ . Then we can show inductively that  $|N_G[I_i]| \leq (d+1)|I_i|$ , where  $N_G[I_i]$  denotes the set of vertices within distance 1 of  $I_i$ . Thus  $I_k$  is the desired independent set.

Consider  $S$  as a graph where  $s$  and  $s'$  are adjacent iff  $s' = sr^{\pm 1}$  and  $\{s, s'\} \neq \{1, r\}$ . Then by the above, there is an independent set  $S' \supseteq \{1, r\}$  of size at least  $\frac{|S|}{3}$ . Fix an injection  $\varphi: T \hookrightarrow 2^{S'}$  such that  $\varphi(t)_1 = \varphi(t)_r = 1$  for every  $t \in T$ ; this is possible since

$$\log_2(|T|) \leq n \log_2(|S|) \leq \frac{|S|}{3} - 2 \leq |S'| - 2$$

by our choice of  $n$ . Define the continuous map  $f: X \rightarrow 2$  by

$$f(x) = \begin{cases} \varphi(x_{s^{-1}})_s & s^{-1} \in \text{Supp}(x) \text{ for some } s \in S' \\ 0 & \text{otherwise} \end{cases}$$

This is well-defined, since if  $s_0^{-1}$  and  $s_1^{-1}$  are both in  $\text{Supp}(x)$ , then since  $\text{Supp}(x)$  is right  $S^3$ -disjoint, we have  $s_0 = s_1$ .

We claim that

$$\gamma \in \text{Supp}(x) \iff f(\gamma^{-1} \cdot x) = f((\gamma r)^{-1} \cdot x) = 1$$

For ( $\implies$ ), since  $x_\gamma \in T$ , we have  $\varphi(x_\gamma)_1 = \varphi(x_\gamma)_r = 1$ , which is equivalent to what we need. For ( $\impliedby$ ), we must have some  $s_0, s_1 \in S'$  such that  $s_0^{-1} \in \text{Supp}(\gamma^{-1} \cdot x)$  and  $s_1^{-1} \in \text{Supp}((\gamma r)^{-1} \cdot x)$ . Thus  $\gamma s_0^{-1}, \gamma r s_1^{-1} \in \text{Supp}(x)$ , but since  $\text{Supp}(x)$  is right  $S^3$ -disjoint, we get that  $\gamma s_0^{-1} = \gamma r s_1^{-1}$ , i.e.  $s_1 = s_0 r$ . Thus by our choice of  $S'$ , we

have  $\{s_0, s_1\} = \{1, r\}$ , and since  $r^2 \neq 1$ , we have  $s_0 = 1$ . and thus  $1 \in \text{Supp}(\gamma^{-1} \cdot x)$ , i.e.  $\gamma \in \text{Supp}(x)$ .

We now show that  $f$  is a generator. Let  $x, x' \in X$ , and suppose that  $f(\gamma \cdot x) = f(\gamma \cdot x')$  for every  $\gamma \in \Gamma$ . Then by above, we have that  $\text{Supp}(x) = \text{Supp}(x')$ . If  $\gamma \notin \text{Supp}(x)$ , then  $x_\gamma = * = x_{\gamma'}$ . If  $\gamma \in \text{Supp}(x)$ , then for any  $s \in S'$ , we have

$$\varphi(x_\gamma)_s = f((\gamma s)^{-1} \cdot x) = f((\gamma s)^{-1} \cdot x') = \varphi(x'_\gamma)_s$$

So since  $\varphi$  is injective, we have  $x_\gamma = x'_\gamma$ . Thus  $x = x'$ , and  $f$  is a generator.  $\square$

It turns out that if one is willing to replace  $2^\Gamma$  by  $k^\Gamma$ , where  $k$  depends on  $\Gamma$ , it is easier to get compressible subshifts.

Fix a group  $\Gamma$ . For finite subsets  $S$  and  $T$  of  $\Gamma$ , denote by  $X_{S,T}$  the space of  $(S, T)$ -paradoxical decompositions of  $\Gamma$ , that is, the subshift of  $(S \sqcup T)^\Gamma$  such that  $x \in X_{S,T}$  iff  $\{x^{-1}(s)s\}_{s \in S}$  and  $\{x^{-1}(t)t\}_{t \in T}$  are both partitions of  $\Gamma$  (we allow pieces of a partition to be empty).

For a finite subset  $T$  of  $\Gamma$ , denote by  $X_T$  be the space of 2-to-1  $T$ -surjections of  $\Gamma$ , that is, the subshift of  $T^\Gamma$  such that  $x \in X_T$  iff the map  $\Gamma \rightarrow \Gamma$  defined by  $\gamma \mapsto \gamma x_\gamma$  is a 2-to-1 surjection.

Note that  $X_{S,T}$  is a subset of  $X_{S \cup T}$ . Also,  $\Gamma$  is non-amenable iff  $X_{S,T}$  is nonempty for some  $S$  and  $T$  iff  $X_T$  is nonempty for some  $T$ .

The Tarski number  $k_\Gamma$  of  $\Gamma$  is minimum of  $|S| + |T|$  over all  $S$  and  $T$  with  $X_{S,T}$  nonempty (it's the smallest number of pieces in a paradoxical decomposition). There is a number  $l_\Gamma$  which is the minimum of  $|T|$  over all  $T$  with  $X_T$  nonempty, or equivalently, the minimum of  $|S \cup T|$  over all  $S$  and  $T$  with  $X_{S,T}$  nonempty (it's the smallest number of group elements required in a paradoxical decomposition). Note that we have  $l_\Gamma < k_\Gamma$  for any non-amenable  $\Gamma$ , since if  $X_{S,T}$  is nonempty, then  $X_{S,T\gamma}$  is nonempty for any  $\gamma$ , and thus we can assume that  $S$  and  $T$  have at least one element in common, i.e.  $|S \cup T| < |S| + |T|$ . Note that by [EGS15] there are groups  $\Gamma$  with arbitrarily large  $k_\Gamma$ .

**Proposition 3.7.3.**  *$X_{S,T}$  and  $X_T$  are compressible. Thus if  $\Gamma$  is non-amenable, then there is a compressible subshift of  $(l_\Gamma)^\Gamma$ .*

So, for example, this easily gives a compressible subshift of  $3^{\mathbb{F}_2}$ .

*Proof.* For  $X_{S,T}$ , let  $P$  and  $Q$  be the set of  $x \in X_{S,T}$  such that  $x_1 \in S$  and  $x_1 \in T$  respectively. Then the map defined by  $x \mapsto x_1^{-1}x$  is a bijection from  $P \rightarrow X_{S,T}$  and a bijection  $Q \rightarrow X_{S,T}$ , so  $X_{S,T}$  is equidecomposable with two copies of itself, and thus it is compressible.

For  $X_T$ , let  $P$  be the set of  $x \in X_T$  such that  $x_1$  is the least of the two elements of  $\{\gamma : \gamma x_\gamma = x_1\}$  (in some fixed ordering). Then proceed as above.  $\square$

From [Theorem 3.7.1](#) a group  $\Gamma$  is non-amenable iff there is a compressible subshift of  $2^\Gamma$ . The following question asks whether an analogous characterization exists for groups that contain  $\mathbb{F}_2$ .

**Problem 3.7.4.** *Is it true that a group  $\Gamma$  contains  $\mathbb{F}_2$  iff there is a compressible, orbit-universal subshift of  $2^\Gamma$ ?*

### 3.8 The space of subshifts

(A) We will first review the standard embedding of actions into the shift action. Consider a continuous action of a countable group  $\Gamma$  on a Polish space  $Y$  and let  $Y$  be a closed subspace of a Polish space  $X$ . Define  $f: Y \rightarrow X^\Gamma$  by

$$f(y)_\gamma = \gamma^{-1} \cdot y.$$

Then it is easy to check that  $f$  is  $\Gamma$ -equivariant,  $f(Y)$  is a closed subset of  $X^\Gamma$  and  $f$  is a homeomorphism of  $Y$  with  $f(Y)$ , i.e., the action of  $\Gamma$  on  $Y$  is (topologically) isomorphic to a subshift of  $X^\Gamma$ , where of course  $\Gamma$  acts on itself by left translation.

For any Polish space  $X$ , define the standard Borel space of subshifts of  $X^\Gamma$  as follows:

$$\text{Sh}(\Gamma, X) = \{F \in F(X^\Gamma) : F \text{ is } \Gamma\text{-invariant}\}$$

If  $X$  is compact, we view this as a compact Polish space with the Vietoris topology.

Consider the Hilbert cube  $\mathbb{I}^\mathbb{N}$ . Every compact Polish space is (up to homeomorphism) a closed subspace of  $\mathbb{I}^\mathbb{N}$ , and thus every  $\Gamma$ -flow is (topologically) isomorphic to a subshift of  $(\mathbb{I}^\mathbb{N})^\Gamma$ . We can thus consider the compact Polish space  $\text{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$  as the universal space of  $\Gamma$ -flows.

Similarly consider the product space  $\mathbb{R}^\mathbb{N}$ . Every Polish space is (up to homeomorphism) a closed subspace of  $\mathbb{R}^\mathbb{N}$ , and thus every continuous  $\Gamma$ -action is (topologically) isomorphic to a subshift of  $(\mathbb{R}^\mathbb{N})^\Gamma$ . We can thus consider the standard Borel space  $\text{Sh}(\Gamma, \mathbb{R}^\mathbb{N})$  as the universal space of continuous  $\Gamma$ -actions.

In particular taking  $\Gamma = \mathbb{F}_\infty$ , the free group with a countably infinite set of generators, we see that every CBER is Borel isomorphic to the equivalence relation  $E_F$  induced on some subshift  $F$  of  $(\mathbb{R}^\mathbb{N})^{\mathbb{F}_\infty}$  and so we can view  $\text{Sh}(\mathbb{F}_\infty, \mathbb{R}^\mathbb{N})$  also as the universal space of CBER and study the complexity of various classes of CBER (like, e.g., smooth, aperiodic, hyperfinite, etc.) as subsets of this universal space. Similarly we can view  $\text{Sh}(\mathbb{F}_\infty, \mathbb{I}^\mathbb{N})$  as the universal space of CBER that admit a compact action



realization. In this case we can also consider complexity questions as well as generic questions of various classes.

(B) Let  $\Phi$  be a property of continuous  $\Gamma$ -actions on Polish spaces which is invariant under (topological) isomorphism. Let

$$\text{Sh}_\Phi(\Gamma, X) = \{F \in \text{Sh}(\Gamma, X) : F \models \Phi\},$$

where we write  $F \models \Phi$  to mean that  $F$  has the property  $\Phi$ .

Let  $\text{Prob}(\Gamma) := \{p \in \ell^1(\Gamma) : p \geq 0, \|p\|_1 = 1\}$ , the space of probability measures on  $\Gamma$ , viewed as a  $\Gamma$ -space with the action  $[\gamma \cdot p](\delta) := p(\delta\gamma)$ .

A Borel action  $\Gamma \curvearrowright X$  on a standard Borel space is **Borel amenable** if there is a sequence of Borel maps  $p_n : X \rightarrow \text{Prob}(\Gamma)$  such that  $\|p_n^{\gamma \cdot x} - \gamma \cdot p_n^x\|_1 \rightarrow 0$  for every  $\gamma \in \Gamma$  and  $x \in X$ . If  $\mu$  is a Borel probability measure on  $X$ , then  $\Gamma \curvearrowright X$  is  **$\mu$ -amenable** if there is a  $\Gamma$ -invariant  $\mu$ -conull Borel subset of  $X$  where the action is Borel amenable. The action  $\Gamma \curvearrowright X$  is **measure-amenable** if it is  $\mu$ -amenable for every  $\mu$ . By [Theorem A.3.1](#), measure-amenability is equivalent to **topological amenability** (defined in [Appendix A](#)).

A countable discrete group  $\Gamma$  is **exact** if it admits a measure-amenable (equiv. topologically amenable) action on a compact Polish space, in which case there exists such an action on the Cantor space  $2^{\mathbb{N}}$ , since every compact  $\Gamma$ -flow extends to a  $\Gamma$ -flow on  $2^{\mathbb{N}}$ , see [\[GdlH97\]](#).

A CBER  $E$  on  $X$  is  **$\mu$ -hyperfinite** if there is an invariant  $\mu$ -conull Borel subset of  $X$  where the action is hyperfinite and it is **measure-hyperfinite** if it is  $\mu$ -hyperfinite with respect to every  $\mu$ .

A CBER  $E$  on  $X$  is **amenable** if there is a sequence  $p_n : E \rightarrow [0, 1]$  of Borel functions such that  $p_n^x \in \text{Prob}([x]_E)$  for every  $x \in X$ , and for every  $(x, y) \in E$ , we have  $\|p_n^x - p_n^y\|_1 \rightarrow 0$  in  $\ell^1([x]_E)$ . Analogously we define what it means to say that  $E$  is  **$\mu$ -amenable** and **measure-amenable**. By the Connes-Feldman-Weiss theorem, see, e.g., [\[KM04, Theorem 10.1\]](#),  $E$  is  $\mu$ -amenable iff  $E$  is  $\mu$ -hyperfinite and thus  $E$  is measure-amenable iff it is measure-hyperfinite.

We will consider below the following  $\Phi$ :

- 1) fin: finite equivalence relation;
- 2) sm: smooth equivalence relation;
- 3) free: free action;
- 4) aper: aperiodic equivalence relation;

- 5) comp: compressible equivalence relation;
- 6) hyp: hyperfinite equivalence relation;
- 7) amen: amenable equivalence relation;
- 8) measHyp: measure-hyperfinite equivalence relation
- 9) freeMeasHyp: free action + measure-hyperfinite equivalence relation;
- 10) measAmen: measure-amenable action.

We summarize in the following table what we can prove concerning the descriptive or generic properties of the  $\Phi$  above:

| $\Phi$      | $\text{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$ | $\text{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N})$ |                              |
|-------------|---|---|------------------------------|
| fin         | meager  | $\mathbf{\Pi}_1^1$ -complete                    |                              |
| sm          |   |   |                              |
| free        | comeager  | $G_\delta$                                      | $\mathbf{\Pi}_1^1$ -complete |
| aper        |   |   |                              |
| comp        |   | open  |                              |
| hyp         | ?   | $\Sigma_2^1, \mathbf{\Pi}_1^1$ -hard            |                              |
| amen        | comeager  | $\mathbf{\Pi}_1^1$ -complete                    |                              |
| measHyp     |   |   |                              |
| freeMeasHyp |   | $G_\delta$                                      | $\mathbf{\Pi}_1^1$ -complete |
| measAmen    |   |   |                              |

In this table,  $\Gamma$  is an infinite group,  $\Gamma$  is residually finite in the “ $\mathbf{\Pi}_1^1$ -complete” entry of the first two rows,  $\Gamma$  is non-amenable in the “comeager” entry of the fifth row,  $\Gamma$  is non-amenable and residually finite in the “ $\mathbf{\Pi}_1^1$ -hard” and “ $\mathbf{\Pi}_1^1$ -complete” entries of the last five rows, and  $\Gamma$  is exact in the “comeager” entry of the last four rows.

The following two problems are open:

**Problem 3.8.1.** *Let  $\Gamma$  be an infinite group. Is  $\text{Sh}_{hyp}(\Gamma, \mathbb{I}^\mathbb{N})$  comeager in  $\text{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$ ?*

**Problem 3.8.2.** *Let  $\Gamma$  be an infinite group. What is the exact descriptive complexity of  $\text{Sh}_{hyp}(\Gamma, \mathbb{I}^\mathbb{N})$  in  $\text{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$ ?*

Note that from the results in the 5th row, it follows that a countable group  $\Gamma$  is amenable iff the generic subshift of  $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$  admits an invariant probability Borel measure.

We will now prove the results in the table in a series of propositions. A property  $\Phi$  of continuous actions of  $\Gamma$  on Polish spaces, invariant under topological isomorphism is:

- **satisfiable** if some Polish  $\Gamma$ -space satisfies  $\Phi$ ;
- **compactly satisfiable** if some (compact)  $\Gamma$ -flow satisfies  $\Phi$ ;
- **product-stable** if for any Polish  $\Gamma$ -spaces  $\mathbf{a}$  and  $\mathbf{b}$ , if  $\mathbf{a}$  satisfies  $\Phi$ , then  $\mathbf{a} \times \mathbf{b}$  satisfies  $\Phi$ .

**Proposition 3.8.3.** *Let  $\Phi$  be a compactly satisfiable, product-stable property. Then the set*

$$\{K \in \text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}}) : K \text{ satisfies } \Phi\}$$

*is dense in  $\text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ .*

*Proof.* Since  $\mathbb{I}^{\mathbb{N}}$  is the inverse limit of the spaces  $\mathbb{I}^n$ , we have that  $\text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$  is the inverse limit of  $(\text{Sh}(\Gamma, \mathbb{I}^n))_n$ . Thus it suffices to show, for every  $n \in \mathbb{N}$  and every nonempty open  $U \subseteq \text{Sh}(\Gamma, \mathbb{I}^n)$ , that some subshift in  $\pi_n^{-1}(U)$  satisfies  $\Phi$ , where  $\pi_n : \text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}}) \rightarrow \text{Sh}(\Gamma, \mathbb{I}^n)$  is the projection map. Fix  $K \in U$ , and fix  $L \in \text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N} \setminus n})$  satisfying  $\Phi$ . Then  $K \times L$  satisfies  $\Phi$  by product stability, and is contained in  $\pi_n^{-1}(U)$ , so we are done.  $\square$

For compact Polish  $X$ , a subset  $\mathcal{J} \subseteq \text{Sh}(\Gamma, X)$  is a  **$\sigma$ -ideal** if the following hold:

- i) if  $K \in \mathcal{J}$ ,  $L \in \text{Sh}(\Gamma, X)$  and  $L \subseteq K$ , then  $L \in \mathcal{J}$ ;
- ii) if  $K \in \text{Sh}(\Gamma, X)$  and  $K = \bigcup_n K_n$  for some countable sequence  $K_n \in \mathcal{J}$ , then  $K \in \mathcal{J}$ .

Every  $\text{Sh}_{\Phi}(\Gamma, X)$  in the above table is a  $\sigma$ -ideal. We will need the following to show  $\mathbf{\Pi}_1^1$ -hardness. It is an analog of [KLW87, Section 1.4, Theorem 7] and can be proved by the same argument which we repeat here for the convenience of the reader.

**Proposition 3.8.4.** *Let  $X$  be a compact Polish space and let  $\mathcal{J}$  be a  $\sigma$ -ideal in  $\text{Sh}(\Gamma, X)$ . If  $\mathcal{J}$  is  $F_{\sigma}$ -hard, then  $\mathcal{J}$  is  $\mathbf{\Pi}_1^1$ -hard.*

*Proof.* There is a continuous map  $2^{\mathbb{N}} \rightarrow \text{Sh}(\Gamma, X)$  reducing  $2^{<\mathbb{N}} \subseteq 2^{\mathbb{N}}$  to  $\mathcal{J}$ , which we will denote by  $\alpha \mapsto K_\alpha$ . Then the continuous map  $K(2^{\mathbb{N}}) \rightarrow \text{Sh}(\Gamma, X)$  defined by  $A \mapsto \bigcup_{\alpha \in A} K_\alpha$  reduces  $K(2^{<\mathbb{N}}) = \{K \in K(2^{\mathbb{N}}) : K \subseteq 2^{<\mathbb{N}}\}$  to  $\mathcal{J}$ , since for every  $A \in K(2^{\mathbb{N}})$ , we have

$$\begin{aligned} A \subseteq 2^{<\mathbb{N}} &\implies K_\alpha \in \mathcal{J} \text{ for all } \alpha \in A, \text{ and } A \text{ is countable} \\ &\implies \bigcup_{\alpha \in A} K_\alpha \in \mathcal{J} \\ &\implies K_\alpha \in \mathcal{J} \text{ for all } \alpha \in A \\ &\implies A \subseteq 2^{<\mathbb{N}}. \end{aligned}$$

So the result follows, since  $K(2^{<\mathbb{N}})$  is  $\mathbf{\Pi}_1^1$ -hard (see [Kec95, 27.4(ii)]).  $\square$

For a subset  $(F_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$  of  $\text{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$ , there is a closed  $\Gamma$ -invariant subspace of  $\mathbb{N}^{\mathbb{N}} \times ((\mathbb{R}^{\mathbb{N}})^\Gamma)^{\mathbb{N}}$  given by

$$\prod_{\alpha \in \mathbb{N}^{\mathbb{N}}} \prod_n F_{\alpha|n} = \{(\alpha, (x_n)_n) \in \mathbb{N}^{\mathbb{N}} \times ((\mathbb{R}^{\mathbb{N}})^\Gamma)^{\mathbb{N}} : \forall n [x_n \in F_{\alpha|n}]\}.$$

Fixing a closed embedding  $\mathbb{N}^{\mathbb{N}} \times ((\mathbb{R}^{\mathbb{N}})^\Gamma)^{\mathbb{N}} \hookrightarrow \mathbb{R}^{\mathbb{N}}$ , we obtain an element of  $\text{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$ , which we denote by  $\mathcal{A}_s F_s$ .

**Proposition 3.8.5.** *Let  $\Phi$  and  $\Psi$  be disjoint satisfiable properties of Polish  $\Gamma$ -spaces such that*

- i) if  $(F_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$  is a subset of  $\text{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$  such that  $\{s \in \mathbb{N}^{<\mathbb{N}} : F_s \not\models \Phi\}$  is well-founded, then  $\mathcal{A}_s F_s$  satisfies  $\Phi$ ;*
- ii) if  $(F_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$  is a subset of  $\text{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$  such that  $\{s \in \mathbb{N}^{<\mathbb{N}} : F_s \models \Psi\}$  is ill-founded, then  $\mathcal{A}_s F_s$  satisfies  $\Psi$ .*

*Then  $\text{Sh}_\Phi(\Gamma, \mathbb{R}^{\mathbb{N}})$  is  $\mathbf{\Pi}_1^1$ -hard.*

*Proof.* Let  $\text{Tr} \subseteq 2^{\mathbb{N}^{<\mathbb{N}}}$  denote the space of trees, and let  $\text{WF} \subseteq \text{Tr}$  be the subset of well-founded trees, which is  $\mathbf{\Pi}_1^1$ -complete; see [Kec95, 33.A].

Fix  $F_\Phi, F_\Psi \in \text{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$  satisfying  $\Phi$  and  $\Psi$ , respectively, and for every  $T \in \text{Tr}$  and  $s \in \mathbb{N}^{<\mathbb{N}}$ , define  $F_s^T \in \text{Sh}_\Phi(\Gamma, \mathbb{R}^{\mathbb{N}})$  by

$$F_s^T := \begin{cases} F_\Phi & s \notin T \\ F_\Psi & s \in T \end{cases}$$

Then

$$\begin{aligned} T \in \text{WF} &\implies \mathcal{A}_s F_s^T \models \Phi \\ T \notin \text{WF} &\implies \mathcal{A}_s F_s^T \models \Psi \end{aligned}$$

so the Borel map  $T \mapsto \mathcal{A}_s F_s^T$  is a reduction from WF to  $\text{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N})$ , whence the latter is  $\mathbf{\Pi}_1^1$ -hard.  $\square$

**Proposition 3.8.6.** *Let  $\Gamma$  be a countably infinite group, and let  $\Phi \in \{\text{free}, \text{aper}\}$ . Then  $\text{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$  is dense  $G_\delta$ , and  $\text{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N})$  is  $\mathbf{\Pi}_1^1$ -complete.*

*Proof.* For every  $\gamma \in \Gamma$ , the set of fixed points of  $\gamma$  in  $(\mathbb{I}^\mathbb{N})^\Gamma$  (resp.,  $(\mathbb{R}^\mathbb{N})^\Gamma$ ) is closed (resp., Borel). Thus the set of points with free orbit is  $G_\delta$  (resp., Borel), whence  $\text{Sh}_{\text{free}}(\Gamma, \mathbb{I}^\mathbb{N})$  is  $G_\delta$  (resp.,  $\text{Sh}_{\text{free}}(\Gamma, \mathbb{R}^\mathbb{N})$  is  $\mathbf{\Pi}_1^1$ ). Similarly, the set of aperiodic points in  $(\mathbb{I}^\mathbb{N})^\Gamma$  (resp.,  $(\mathbb{R}^\mathbb{N})^\Gamma$ ) is  $G_\delta$  (resp., Borel), so  $\text{Sh}_{\text{aper}}(\Gamma, \mathbb{I}^\mathbb{N})$  is  $G_\delta$  (resp.,  $\text{Sh}_{\text{free}}(\Gamma, \mathbb{R}^\mathbb{N})$  is  $\mathbf{\Pi}_1^1$ ).

The property  $\Phi$  is compactly satisfiable (see, e.g. [KPT05, 1(B)]) and product-stable, so density of  $\text{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$  follows from Proposition 3.8.3.

Finally,  $\mathbf{\Pi}_1^1$ -completeness follows from Proposition 3.8.5 by taking  $\Psi$  to be “has a fixed point”.  $\square$

**Proposition 3.8.7.** *Let  $\Gamma$  be a countably infinite group. Then  $\text{Sh}_{\text{comp}}(\Gamma, \mathbb{I}^\mathbb{N})$  is open,  $\text{Sh}_{\text{comp}}(\Gamma, \mathbb{R}^\mathbb{N})$  is  $\mathbf{\Pi}_1^1$ -complete, and if  $\Gamma$  is non-amenable, then the former is dense.*

*Proof.* By Nadkarni’s theorem,  $F$  is non-compressible iff

$$\exists \mu \in P(F) \forall \gamma [\gamma \cdot \mu = \mu],$$

where  $P(F)$  is the set of Borel probability measures on  $F$ , which is a compact Polish space for  $\mathbb{I}^\mathbb{N}$  and a standard Borel space for  $\mathbb{R}^\mathbb{N}$ . Thus the set of compressible subshifts is open for  $\mathbb{I}^\mathbb{N}$ , and  $\mathbf{\Pi}_1^1$  for  $\mathbb{R}^\mathbb{N}$ . Moreover,  $\mathbf{\Pi}_1^1$ -completeness follows from Proposition 3.8.5 by taking  $\Phi$  to be “compressible” and  $\Psi$  to be “has a fixed point”.

Now suppose  $\Gamma$  is non-amenable. Then compressibility is compactly satisfiable by non-amenableity, and it is product-stable, so density follows from Proposition 3.8.3.  $\square$

**Proposition 3.8.8.** *Let  $\Gamma$  be a countably infinite group, let  $X$  be a Polish space, and let  $\Phi \in \{\text{free}, \text{sm}\}$ . Then  $\text{Sh}_\Phi(\Gamma, X)$  is  $\mathbf{\Pi}_1^1$ , and if  $X = \mathbb{I}^\mathbb{N}$ , then it is meager.*

*Proof.* The set of periodic points in  $X^\Gamma$  is Borel. Also, a subshift is smooth iff every orbit is discrete (see, e.g., [Kec10, Corollary 22.3]). The set of points with discrete orbit is Borel. So in either case,  $\text{Sh}_\Phi(\Gamma, X)$  is  $\mathbf{\Pi}_1^1$ .

If  $X = \mathbb{I}^\mathbb{N}$ , then meagerness follows since  $\text{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$  is disjoint from  $\text{Sh}_{\text{aper}}(\Gamma, \mathbb{I}^\mathbb{N})$ , which is comeager by Proposition 3.8.6 (see also here Proposition 3.2.3).  $\square$

We now turn to the various notions of amenability.

**Proposition 3.8.9.** *Let  $\Gamma$  be a countably infinite group and let  $X$  be a Polish space. Then for  $\Phi$  in  $\{\text{hyp}, \text{amen}\}$  or  $\{\text{measHyp}, \text{freeMeasHyp}, \text{measAmen}\}$ ,  $\text{Sh}_\Phi(\Gamma, X)$  is  $\Sigma_2^1$  or  $\Pi_1^1$  respectively. If moreover  $\Gamma$  is non-amenable, then  $\text{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N})$  is  $\Pi_1^1$ -hard.*

*Proof.* First,  $\text{Sh}_{\text{hyp}}(\Gamma, X)$  is  $\Sigma_2^1$ , since  $F$  is hyperfinite iff

$$\begin{aligned} &\exists \text{ sequence } (E_n)_n \text{ of Borel subsets of } (X^\Gamma)^2 \\ &\quad [\forall n [E_n \text{ is a finite equivalence relation and } E_n \subseteq E_{n+1}] \\ &\quad \text{and } \forall x \in F \forall \gamma \exists n [(\gamma \cdot x, x) \in E_n]]. \end{aligned}$$

Next,  $\text{Sh}_{\text{amen}}(\Gamma, X)$  is  $\Sigma_2^1$ , since  $F$  is amenable iff

$$\begin{aligned} &\exists \text{ sequence } (f_n)_n \text{ of Borel functions } f_n : X^2 \rightarrow [0, 1] \\ &\quad \forall x \in F \left[ \forall n \sum_{y \in [x]_{E_\Gamma^X}} f_n^x(y) = 1 \text{ and } \forall y \in [x]_{E_\Gamma^X} \|f_n^x - f_n^y\|_1 \rightarrow 0 \right], \end{aligned}$$

where  $\|\cdot\|_1$  is on  $\ell^1([x]_{E_\Gamma^X})$ .

$\text{Sh}_{\text{measHyp}}(\Gamma, X)$  is  $\Pi_1^1$  by Miri Segal's effective witness to measure-hyperfiniteness (see [CM17, Theorem 1.7.8]).

Now  $\text{Sh}_{\text{freeMeasHyp}}(\Gamma, X)$  is  $\Pi_1^1$ , since  $\text{Sh}_{\text{free}}(\Gamma, X)$  and  $\text{Sh}_{\text{measHyp}}$  are  $\Pi_1^1$ .

Similarly, the set of points with amenable stabilizer is  $G_\delta$ , since  $x$  has amenable stabilizer iff

$$\forall S \in \text{Fin}(\Gamma) \left[ S \subseteq \Gamma_x \implies \forall n \in \mathbb{N} \exists F \subseteq \langle S \rangle \frac{|SF \Delta F|}{|F|} < \frac{1}{n} \right].$$

Thus the set

$$\{F \in \text{Sh}(\Gamma, X) : \forall x \in F [\Gamma_x \text{ is amenable}]\}$$

is  $\Pi_1^1$  (in fact  $G_\delta$  when  $X$  is compact), and thus  $\text{Sh}_{\text{measAmen}}(\Gamma, X)$  is  $\Pi_1^1$  by [Corollary A.2.2](#).

If  $\Gamma$  is non-amenable, then  $\Pi_1^1$ -hardness follows from [Proposition 3.8.5](#) by taking  $\Psi$  to be “has a free non-compressible  $\Gamma$ -invariant closed subspace”.  $\square$

Surprisingly, the free measure-hyperfinite subshifts of  $(\mathbb{I}^\mathbb{N})^{\mathbb{F}^\infty}$  form a  $G_\delta$  set:

**Proposition 3.8.10.** *Let  $\Gamma$  be an infinite exact group. Then  $\text{Sh}_{\text{measAmen}}(\Gamma, \mathbb{I}^\mathbb{N})$  and  $\text{Sh}_{\text{freeMeasHyp}}(\Gamma, \mathbb{I}^\mathbb{N})$  are dense  $G_\delta$ . Moreover  $\text{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$  is comeager for  $\Phi \in \{\text{amen}, \text{measHyp}\}$ .*

*Proof.* Measure-amenability is compactly satisfiable (by exactness) and product-stable, so it is dense by [Proposition 3.8.3](#). To show that  $\text{Sh}_{\text{measAmen}}(\Gamma, X)$  is  $G_\delta$ , by [Proposition 3.8.4](#), it suffices to show that it is  $\Sigma_1^1$ .

We use the characterization of measure-amenability as topological amenability, see [Theorem A.3.1](#). By [[Kec95](#), 12.13], there is Borel function  $D : K(X) \rightarrow X^\mathbb{N}$  such that  $D(K)$  is a dense subset of  $K$  for every nonempty  $K \in K(X)$ , and we can assume that  $D(K)$  is  $\Gamma$ -invariant. Fix a compatible metric  $d$  on  $X$ . Then a subshift  $K$  is topologically amenable iff for every  $\varepsilon > 0$  and any finite  $S \subseteq \Gamma$ , there is a function  $p : \mathbb{N} \rightarrow \text{Prob}(\Gamma)$  such that

- i) (uniform continuity) for every  $\varepsilon_1$ , there is a  $\varepsilon_2$  such that for every  $n, m \in \mathbb{N}$ , if  $d(D(K)_n, D(K)_m) < \varepsilon_2$ , then  $\|p^n - p^m\|_1 < \varepsilon_1$ ;
- ii) (invariance) for every  $\gamma \in S$  and every  $n, m \in \mathbb{N}$ , if  $D(K)_n = \gamma \cdot D(K)_m$ , then  $\|p^n - \gamma \cdot p^m\|_1 < \varepsilon$ .

So it is  $\Sigma_1^1$ .

Now  $\text{Sh}_{\text{freeMeasHyp}}(\Gamma, \mathbb{I}^\mathbb{N})$  is  $G_\delta$ , since by [Corollary A.2.2](#), it is the intersection of  $\text{Sh}_{\text{free}}(\Gamma, \mathbb{I}^\mathbb{N})$  and  $\text{Sh}_{\text{measAmen}}(\Gamma, \mathbb{I}^\mathbb{N})$  which are both dense  $G_\delta$  (the former by [Proposition 3.8.6](#)).

Finally, by the diagram of implications in the beginning of [Appendix A](#), we have that  $\text{Sh}_{\text{freeMeasHyp}}(\Gamma, \mathbb{I}^\mathbb{N}) \subseteq \text{Sh}_{\text{amen}}(\Gamma, \mathbb{I}^\mathbb{N}) \subseteq \text{Sh}_{\text{measHyp}}(\Gamma, \mathbb{I}^\mathbb{N})$ , so the last two classes are also comeager.  $\square$

We conclude by showing  $\Pi_1^1$ -hardness of  $\text{Sh}_\Phi(\mathbb{F}_\infty, \mathbb{I}^\mathbb{N})$  for various  $\Phi$ .

**Proposition 3.8.11.** *Let  $\Gamma$  be an infinite residually finite group, and let  $X$  be  $\mathbb{I}^\mathbb{N}$  or  $\mathbb{R}^\mathbb{N}$ . Then  $\text{Sh}_\Phi(\Gamma, X)$  is  $\Pi_1^1$ -hard, where  $\Phi \in \{\text{fin}, \text{sm}\}$ . If moreover  $\Gamma$  is non-amenable, then  $\text{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$  is  $\Pi_1^1$ -hard, where  $\Phi \in \{\text{hyp}, \text{amen}, \text{measHyp}\}$ .*

*Proof.* Since  $\text{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$  reduces to  $\text{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N})$  via the inclusion map, it suffices to consider the case where  $X = \mathbb{I}^\mathbb{N}$ . By [Proposition 3.8.4](#), it suffices to show  $F_\sigma$ -hardness. We will define a continuous map  $2^\mathbb{N} \rightarrow \text{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$  which simultaneously reduces  $2^{<\mathbb{N}}$  to  $\text{Sh}_{\text{fin}}(\Gamma, \mathbb{I}^\mathbb{N})$  and to  $\text{Sh}_{\text{sm}}(\Gamma, \mathbb{I}^\mathbb{N})$ , and if moreover  $\Gamma$  is also non-amenable also to  $\text{Sh}_{\text{measHyp}}(\Gamma, \mathbb{I}^\mathbb{N})$ . Fix a descending sequence  $(\Lambda_n)_n$  of finite index subgroups of  $\Gamma$  such that  $\bigcap_n \Lambda_n = \{1\}$ .

Let  $K_i^n \in \text{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$ ,  $n \in \mathbb{N}$ ,  $i \in \{0, 1\}$ , be defined as follows:  $K_0^n$  is an invariant singleton and  $K_1^n$  is a subshift isomorphic to the action of  $\Gamma$  on  $\Gamma/\Lambda_n$ . Consider now the space  $\prod_n (\mathbb{I}^\mathbb{N})^\Gamma$  on which  $\Gamma$  acts diagonally and let  $\Phi : \prod_n (\mathbb{I}^\mathbb{N})^\Gamma \rightarrow (\mathbb{I}^\mathbb{N})^\Gamma$  be a  $\Gamma$ -equivariant continuous embedding. Finally for each  $\alpha \in 2^\mathbb{N}$ , let

$$\varphi(\alpha) = \Phi\left(\prod_n K_{\alpha_n}^n\right).$$

Then  $\varphi: 2^{\mathbb{N}} \rightarrow \text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$  is continuous. If  $\alpha \in 2^{<\mathbb{N}}$ , clearly  $\varphi(\alpha)$  is finite. If  $\alpha \notin 2^{<\mathbb{N}}$ , then  $\varphi(\alpha)$  is a free subshift admitting an invariant probability Borel measure, so it is not smooth. If moreover  $\Gamma$  is non-amenable, it is also not measure-hyperfinite.  $\square$

The preceding complexity calculations have some relevance to the question of whether every non-smooth, aperiodic CBER admits a compact action realization.

**Proposition 3.8.12.** *For every  $x \in 2^{\mathbb{N}}$ , there is a non-smooth, aperiodic  $F \in \text{Sh}(\mathbb{F}_{\infty}, \mathbb{R}^{\mathbb{N}})$  such that there is no  $\Delta_1^1(F, x)$  isomorphism of  $E_F$  with some  $E_K$ ,  $K \in \text{Sh}(\mathbb{F}_{\infty}, \mathbb{I}^{\mathbb{N}})$ .*

*Proof.* Assume this fails toward a contradiction. Then there is a  $\Pi_1^1$  definition of the class of all  $F \in \text{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$  that are aperiodic and  $E_F$  admits a compact action realization. Now for each  $K \in \text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ , we have that

$$K \notin \text{Sh}_{\text{sm}}(\Gamma, \mathbb{I}^{\mathbb{N}}) \iff K \times I_{\mathbb{N}} \text{ admits a compact action realization,}$$

so the class  $\text{Sh}_{\text{sm}}(\Gamma, \mathbb{I}^{\mathbb{N}})$  is  $\Sigma_1^1$ , contradicting [Proposition 3.8.11](#).  $\square$

Informally this implies that there is no “uniform Borel method” that will construct a compact action realization for each aperiodic, non-smooth CBER, even if it is given as a subshift of  $(\mathbb{R}^{\mathbb{N}})^{\mathbb{F}_{\infty}}$ .

(C) Let  $\Gamma$  be a countable group, and let  $X$  be a compact zero-dimensional Polish space. Denote by  $\text{Act}(\Gamma, X)$  the space of group homomorphisms  $\Gamma \rightarrow \text{Homeo}(X)$ , i.e.,  $\Gamma$ -flows on the space  $X$ . For  $\mathbf{a} \in \text{Act}(\Gamma, X)$ , let  $\mathcal{A}_{\mathbf{a}}$  denote the Boolean algebra of clopen  $\mathbf{a}$ -invariant subsets of  $X$ , and let  $\text{St}(\mathcal{A}_{\mathbf{a}})$  denote its Stone space. There is a continuous  $\mathbf{a}$ -invariant surjection  $2^{\mathbb{N}} \rightarrow \text{St}(\mathcal{A}_{\mathbf{a}})$  defined by sending  $x$  to the ultrafilter  $\{A \in \mathcal{A}_{\mathbf{a}} : x \in A\}$ . For every  $\mathcal{U} \in \text{St}(\mathcal{A}_{\mathbf{a}})$ , the fiber  $C_{\mathcal{U}}^{\mathbf{a}}$  above  $\mathcal{U}$  is a closed  $\mathbf{a}$ -invariant subset of  $X$ , giving the decomposition

$$X = \bigsqcup_{\mathcal{U} \in \text{St}(\mathcal{A}_{\mathbf{a}})} C_{\mathcal{U}}^{\mathbf{a}}.$$

Let  $\text{CEINV}(\mathbf{a})$  denote the subset of the space  $\text{INV}(\mathbf{a})$  of invariant probability Borel measures for  $\mathbf{a}$ , consisting of **clopen-ergodic** measures, that is, measures  $\mu \in \text{INV}(\mathbf{a})$  for which every  $A \in \mathcal{A}_{\mathbf{a}}$  is  $\mu$ -null or  $\mu$ -conull. Note that  $\text{CEINV}(\mathbf{a})$  is closed by the Portmanteau Theorem [[Kec95](#), 17.20(v)], and we have

$$\text{EINV}(\mathbf{a}) \subseteq \text{CEINV}(\mathbf{a}) \subseteq \text{INV}(\mathbf{a}).$$



There is a surjection  $\text{CEINV}(\mathbf{a}) \rightarrow \text{St}(\mathcal{A}_{\mathbf{a}})$  sending  $\mu$  to the ultrafilter  $\{A \in \mathcal{A}_{\mathbf{a}} : \mu(A) = 1\}$ , and the fiber above  $\mathcal{U}$  can be identified with  $\text{INV}(\mathbf{a}|C_{\mathcal{U}}^{\mathbf{a}})$ , giving a decomposition

$$\text{CEINV}(\mathbf{a}) = \bigsqcup_{\mathcal{U} \in \text{St}(\mathcal{A}_{\mathbf{a}})} \text{INV}(\mathbf{a}|C_{\mathcal{U}}^{\mathbf{a}}).$$

**Proposition 3.8.13.** *Suppose  $\Gamma$  is amenable. Let  $\mathbf{a} \in \text{Act}(\Gamma, X)$ . If  $|\mathcal{A}_{\mathbf{a}}| > 2$ , then  $\text{CEINV}(\mathbf{a})$  is a proper subset of  $\text{INV}(\mathbf{a})$ , so in particular,  $\text{INV}(\mathbf{a})$  is not Poulsen. If  $\mathcal{A}_{\mathbf{a}}$  is atomless, then  $\text{EINV}(\mathbf{a})$  has size continuum.*

*Proof.* If  $|\mathcal{A}_{\mathbf{a}}| > 2$ , then  $|\text{St}(\mathcal{A}_{\mathbf{a}})| \geq 2$ , so we see from the decomposition that  $\text{CEINV}(\mathbf{a})$  is not closed under convex combinations, and is thus a strict subset of  $\text{INV}(\mathbf{a})$ . If  $\mathcal{A}_{\mathbf{a}}$  is atomless, then  $\text{St}(\mathcal{A}_{\mathbf{a}})$  has size continuum, so  $\text{EINV}(\mathbf{a})$  has size continuum, since each  $\text{INV}(\mathbf{a}|C_{\mathcal{U}}^{\mathbf{a}})$  is nonempty by amenability of  $\Gamma$ .  $\square$

The following fact was pointed out by J. Melleray (this is also [Ele19, Remark 5]):

**Proposition 3.8.14.** *Consider the action of  $\text{Homeo}(2^{\mathbb{N}})$  by conjugation on  $\text{Act}(\Gamma, 2^{\mathbb{N}})$ . Then there is a dense conjugacy class.*

*Proof.* Let  $(\mathbf{a}_n)$  be a dense sequence in  $\text{Act}(\Gamma, 2^{\mathbb{N}})$  and consider the product action  $\prod_n \mathbf{a}_n$ . Then an isomorphic copy of this action in  $\text{Act}(\Gamma, 2^{\mathbb{N}})$  has dense conjugacy class.  $\square$

**Proposition 3.8.15.** *Suppose  $\Gamma$  is finitely generated. Then for comeagerly many  $\mathbf{a} \in \text{Act}(\Gamma, 2^{\mathbb{N}})$ ,  $\mathcal{A}_{\mathbf{a}}$  is atomless, so in particular if  $\Gamma$  is amenable, then  $\text{EINV}(\mathbf{a})$  has size continuum and  $\text{INV}(\mathbf{a})$  is not Poulsen.*

*Proof.* Let  $\mathcal{A}$  be the set of nonempty clopen subsets of  $2^{\mathbb{N}}$ . Then  $\mathcal{A}_{\mathbf{a}}$  is atomless iff for every  $A \in \mathcal{A}$ , if  $A$  is  $\mathbf{a}$ -invariant, then there is a partition  $A = A_0 \sqcup A_1$  into  $\mathbf{a}$ -invariant  $A_0, A_1 \in \mathcal{A}$ . So it suffices to fix  $A \in \mathcal{A}$ , and show comeagerness of the set of  $\mathbf{a}$  such that if  $A$  is  $\mathbf{a}$ -invariant, then there is a partition  $A = A_0 \sqcup A_1$  into  $\mathbf{a}$ -invariant  $A_0, A_1 \in \mathcal{A}$ . This set is open, since  $\Gamma$  is finitely generated, so it suffices to show that it is dense. Let  $V$  be a nonempty open subset of  $\text{Act}(\Gamma, 2^{\mathbb{N}})$ . We can assume that  $A$  is  $\mathbf{a}$ -invariant for every  $\mathbf{a} \in V$ , otherwise we are done. Then  $V$  gives by restriction an open subset of  $\text{Act}(\Gamma, A)$ , so since the set of  $\mathbf{a} \in \text{Act}(\Gamma, A)$  with a partition  $A = A_0 \sqcup A_1$  into  $\mathbf{a}$ -invariant sets is closed under conjugation, we are done, since  $\text{Act}(\Gamma, A)$  has a dense conjugacy class (because  $A \cong 2^{\mathbb{N}}$ ).  $\square$

**Problem 3.8.16.** *If  $\Gamma$  is finitely generated and amenable, is it true that for comeagerly many  $\mathbf{a} \in \text{Act}(\Gamma, 2^{\mathbb{N}})$ ,  $\text{INV}(\mathbf{a})$  is a Bauer simplex, i.e.,  $\text{EINV}(\mathbf{a})$  is closed in  $\text{INV}(\mathbf{a})$ ?*

By the Correspondence Theorem of Hochman [Hoc08, Theorem 1.3 and Section 10], it follows that for any amenable, finitely generated  $\Gamma$  the generic subshift of  $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$  admits continuum many ergodic invariant measures and thus if all Borel actions of  $\Gamma$  generate hyperfinite equivalence relations (e.g., if  $\Gamma$  is nilpotent), then the generic subshift of  $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$  gives an equivalence relation Borel isomorphic to  $\mathbb{R}E_0$ .

As we indicated before, we do not know if for nonamenable  $\Gamma$  the generic subshift in  $\text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$  is hyperfinite. If that was the case, since the generic subshift of  $\text{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$  is compressible, it would follow that the (equivalence relation of the) generic subshift would be Borel isomorphic to  $E_t$ .

Of particular interest is the case  $\Gamma = \mathbb{F}_2$ . By the result of Kwiatkowska in [Kwi12] there is a generic action of  $\mathbb{F}_2$  on  $2^{\mathbb{N}}$ , i.e., an action  $\mathbf{a} \in \text{Act}(\mathbb{F}_2, 2^{\mathbb{N}})$  with comeager conjugacy class. Then by the above Correspondence Theorem the question of genericity of hyperfiniteness in  $\text{Sh}(\mathbb{F}_2, \mathbb{I}^{\mathbb{N}})$  is equivalent to following question:

**Problem 3.8.17.** *Let  $\mathbf{a} \in \text{Act}(\mathbb{F}_2, 2^{\mathbb{N}})$  be the generic action of  $\mathbb{F}_2$  on  $2^{\mathbb{N}}$ . Is the equivalence relation  $E_{\mathbf{a}}$  hyperfinite?*

Note that by Proposition 3.8.10 and the Correspondence Principle, this equivalence relation  $E_{\mathbf{a}}$  is amenable.

### 3.9 $K_{\sigma}$ and $F_{\sigma}$ realizations

Clinton Conley raised the following question: Does every aperiodic CBER have a realization as a  $K_{\sigma}$  equivalence relation in a Polish space? We answer this question in the affirmative:

**Theorem 3.9.1.** *Every  $E \in \mathcal{AE}$  has a transitive  $K_{\sigma}$  realization in the Cantor space  $2^{\mathbb{N}}$ .*

*Proof.* Let  $Q = 2^{<\mathbb{N}} \subseteq 2^{\mathbb{N}}$ , and let  $N = 2^{\mathbb{N}} \setminus Q$ . Then  $N$  is homeomorphic to Baire space, so by Proposition 3.2.1, we can assume that  $E = E_{\Gamma}^N$ , where  $\Gamma \curvearrowright N$  is a continuous action of a countable group  $\Gamma$  on  $N$ . For each  $\gamma \in \Gamma$ , let  $R_{\gamma}$  be the relation on  $N$  defined by  $x R_{\gamma} y \iff y = \gamma \cdot x$ . Let  $\overline{R_{\gamma}}$  denote the closure of  $R_{\gamma}$  in  $(2^{\mathbb{N}})^2$ . We claim that  $\overline{R_{\gamma}} \subseteq R_{\gamma} \oplus I_Q$ . Let  $(x, y) \in \overline{R_{\gamma}}$ , and suppose that  $x \in N$  (the case  $y \in N$  is identical). Then there is a sequence  $(x_n, y_n)_n$  in  $R_{\gamma}$  converging to  $(x, y)$ . Since  $x \in N$ , we have that  $y_n = \gamma \cdot x_n \rightarrow \gamma \cdot x$ , so  $y = \gamma \cdot x$ , and thus  $(x, y) \in R_{\gamma}$ , proving the claim. Thus the relation  $E \oplus I_Q$  on  $2^{\mathbb{N}}$  (which is isomorphic to  $E$ ) is equal to  $I_Q \cup \bigcup_{\gamma} \overline{R_{\gamma}}$ , so it is  $K_{\sigma}$ , and it has the dense class  $Q$ .  $\square$

We can ask about  $K_{\sigma}$  and  $F_{\sigma}$  realizations which are minimal. There is a known obstruction, due to Solecki:

**Theorem 3.9.2** (Solecki, [Sol02, Corollary 3.2]). *Every minimal  $K_\sigma$  equivalence relation on a Polish space with at least two classes is not smooth.*

It is open whether this is the only obstruction:

**Problem 3.9.3.** *Let  $E$  be an aperiodic CBER. Does  $E$  have a minimal  $F_\sigma$  realization? If  $E$  is non-smooth, does  $E$  have a minimal  $K_\sigma$  realization?*

We do not even know if an aperiodic smooth CBER has a minimal  $F_\sigma$  realization. [Theorem 3.2.6](#) shows that all non-smooth relations in  $\mathcal{AH}$  have minimal  $K_\sigma$  realizations.

In contrast to Solecki's result, one can show the following:

**Proposition 3.9.4.** *Every aperiodic smooth CBER can be realized as a minimal equivalence relation which is a Boolean combination of  $K_\sigma$  relations in a compact Polish space.*

*Proof.* Here are two such realizations:

1. Consider the equivalence relation  $E_0$  in  $2^\mathbb{N}$ . Let  $A$  be a Cantor set in  $2^\mathbb{N}$  which is a partial transversal for  $E_0$ . Let  $B$  be the  $E_0$ -saturation of  $A$  and put  $Y = 2^\mathbb{N} \setminus B$ . Then  $Y$  is  $G_\delta$ , so a zero-dimensional Polish space (in the relative topology). Every compact subset of  $Y$  has empty interior in  $Y$ , so  $Y$  is homeomorphic to the Baire space  $\mathbb{N}^\mathbb{N}$  (see [Kec95, 7.7]). Therefore there is a continuous bijection  $f: Y \rightarrow A$  (see [Kec95, 7.15]). Let  $F$  be the equivalence relation on  $2^\mathbb{N}$  obtained by adding to each  $E_0$  class  $[a]_{E_0}$ , with  $a \in A$ , the point  $f^{-1}(a)$ . Then  $F$  is smooth with all classes dense. Put

$$S(x, y) \iff x \in B \ \& \ y \in Y \ \& \ \exists z \in A (xE_0z \ \& \ f(y) = z)$$

and

$$T(x, y) \iff S(y, x).$$

Then each of  $S, T$  is the intersection of two  $K_\sigma$  relations with a  $G_\delta$  relation and

$$xFy \iff (x, y \in B \ \& \ xE_0y) \vee S(x, y) \vee T(x, y),$$

so  $F$  is a Boolean combination of  $K_\sigma$  relations as well.

2. Let  $X = \prod_{n \geq 1} 2^n$ , where  $2^n$  is the set of binary sequences of length  $n$ . Let  $Y = \{(x_n) \in X : \exists m \forall n \geq m (x_n \prec x_{n+1})\}$ , and define  $f: X \rightarrow 2^\mathbb{N}$  as follows:

$$f(x) = \begin{cases} \lim_n x_n & x \in Y \\ x_1 \hat{\ } x_2 \hat{\ } x_3 \hat{\ } \cdots & x \notin Y \end{cases}$$

Let  $xEy \iff f(x) = f(y)$ . Then  $E$  is a smooth CBER with all classes dense, and it is easy to check that  $E = F_1 \cup F_2 \cup F_3 \cup F_4$ , where  $F_1$  is  $K_\sigma$ ,  $F_2$  and  $F_3$  are intersections of a  $K_\sigma$  and a  $G_\delta$  relation and  $F_4$  is the equality relation on  $X$ .

□

We next discuss a sharper notion of  $K_\sigma$  realization. Let  $X$  be a compact Polish space and  $E$  a CBER in  $X$ . Recall that we say that  $E$  is **compactly graphable** if there is a compact graphing of  $E$ , i.e., a compact graph (irreflexive, symmetric relation)  $K \subseteq E$  so that the  $E$ -classes are the connected components of  $K$ . Note then that  $E$  is  $K_\sigma$ . A CBER  $E$  has a **compactly graphable realization** if it is Borel isomorphic to a compactly graphable CBER. Clearly every CBER that has a compact action realization implemented by a free continuous action of a *finitely generated* group has a compactly graphable realization. Also clearly a CBER that has a compactly graphable realization admits a  $K_\sigma$  realization.

We now have the following result:

**Theorem 3.9.5.** (a) *Every aperiodic hyperfinite CBER has a compactly graphable realization.*

(B) *Every compressible CBER has a compactly graphable realization.*

*Proof.* (a) This follows from [Theorem 3.2.6](#) for non-compressible hyperfinite CBER. The compressible case is covered in (b).

(b) The proof is a modification of the proof of [Theorem 3.9.1](#). Let  $E$  be a compressible CBER. Then by [[DJK94](#), Proposition 1.8], [[Kec21b](#), Proposition 3.27] and the arguments in the proof of [Proposition 3.2.1](#), we can assume that  $E$  is of the form  $E = E_{\mathbb{F}_2}^N$ , where  $N$  is as in the proof of [Theorem 3.9.1](#). Let  $\alpha, \beta$  be free generators of  $\mathbb{F}_2$  and let  $S$  consist of these generators and their inverses. Finally, as in the proof of [Theorem 3.9.1](#), let  $K = \bigcup_{\gamma \in S} \overline{R_\gamma}$ , and note that if  $F$  is the equivalence relation generated by  $K$  (i.e., the smallest equivalence relation containing  $K$ ), then  $F$  is of the form  $E \oplus R$ , where  $R$  is an equivalence relation on the space  $Q$ . Thus  $E$  is Borel bireducible to  $F$ . Now let  $Y = \{1, 1/2, 1/3, \dots, 0\}$  and define on  $Y$  the compact, connected graph  $R$  given by:

$$yRy' \iff (y = 1 \text{ and } y' \leq 1/2) \text{ or } (y' = 1 \text{ and } y \leq 1/2).$$

Consider now the equivalence relation  $G = F \times I_Y$  on  $2^{\mathbb{N}} \times Y$ , where as usual  $I_Y = Y^2$ . Thus  $(x, y)G(x', y') \iff xFx'$ . Then the compact relation  $\tilde{K}$  on  $2^{\mathbb{N}} \times Y$  given by

$$(x, y)\tilde{K}(x', y') \iff xKx' \text{ and } yRy',$$

is a compact graphing of  $G$ . But  $G$  is Borel bireducible to  $F$  and thus to  $E$ . Since both  $E$  and  $G$  are compressible, they are Borel isomorphic by [Kec21b, Proposition 3.27] and the proof is complete.  $\square$

The following is an open problem:

**Problem 3.9.6.** *Does every aperiodic CBER admit a compactly graphable realization?*

### 3.10 A $\sigma$ -ideal associated to a $K_\sigma$ countable Borel equivalence relation

Suppose that  $X$  is an (uncountable) Polish space and  $E$  a CBER on  $X$ . Denote by  $K(X)$  the space of compact subsets of  $X$  with the usual Vietoris topology (see [Kec95, 4.F]). Let

$$I_E = \{K \in K(X) : [K]_E \neq X\}.$$

Recall that a  $\sigma$  ideal of compact sets is a nonempty subset  $I \subseteq K(X)$  such that  $K \subseteq L \in I \implies K \in I$  (i.e., it is hereditary) and  $K \in K(X), K = \bigcup_n K_n, K_n \in I, \forall n \implies K \in I$  (i.e., it is closed under countable unions which are compact).

**Proposition 3.10.1.** *Let  $X$  be a Polish space and  $E$  a  $K_\sigma$  CBER on  $X$  with all  $E$ -classes dense. Then  $I_E$  is a  $G_\delta$   $\sigma$ -ideal of compact sets.*

*Proof.* Here and in the sequel, notice that since  $E$  is  $K_\sigma$ ,  $X = \{x \in X : (x, x) \in E\}$  (and  $X^2$ ) is also  $K_\sigma$  and  $F_\sigma = K_\sigma$  on  $X$  (and  $X^2$ ).

Clearly  $I_E$  is hereditary. To check closure under countable unions, we will actually show that if  $K_n \in I_E, \forall n$ , then  $[\bigcup_n K_n]_E \neq X$ . Notice that because  $E$  is  $K_\sigma$ , for each compact  $K$  the set  $[K]_E$  is also  $K_\sigma$  and thus if  $K \in I_E$ , then  $X \setminus [K]_E$  is dense  $G_\delta$ . So if  $K_n \in I_E, \forall n$ , and  $[\bigcup_n K_n]_E = \bigcup_n [K_n]_E = X$  this contradicts the Baire Category Theorem. Since

$$K \in I_E \iff \exists x \forall y (y \in K \implies \neg xEy),$$

clearly  $I_E$  is  $\Sigma_1^1$ , thus by [KLW87, Theorem 11] (see also [MZ07, Theorem 1.4]) it is  $G_\delta$ .  $\square$

**Corollary 3.10.2.** *If  $X, E$  are as in Proposition 3.10.1 and moreover  $E$  admits a meager complete section, then  $E$  admits a nowhere dense, compact complete section.*

*Proof.* We have a sequence  $K_n$  of nowhere dense compact sets with  $[\bigcup_n K_n]_E = \bigcup_n [K_n]_E = X$ . Thus for some  $n$ ,  $K_n \notin I_E$ , so  $K_n$  is a nowhere dense, compact complete section.  $\square$

Below denote by  $K_{\aleph_0}(X)$  the  $\sigma$ -ideal of countable compact subsets of  $X$  and by  $\text{MGR}(X)$  the  $\sigma$ -ideal of nowhere dense (i.e., meager) compact subsets of  $X$ .

**Corollary 3.10.3.** *If  $X, E$  are as in [Corollary 3.10.2](#), then*

$$K_{\aleph_0}(X) \subsetneq I_E \subsetneq \text{MGR}(X).$$

**Corollary 3.10.4.** *If  $X, E$  are as in [Proposition 3.10.1](#), then  $E$  does not admit a  $K_\sigma$  transversal.*

*Proof.* If  $F$  is a  $K_\sigma$  transversal, we can write  $F = F_1 \sqcup F_2$ , where each  $F_i$  is also  $K_\sigma$  and nonempty. Then each  $F_i$  is the union of countably many compact sets in  $I_E$ , a contradiction.  $\square$

We say that a  $\sigma$ -ideal of compact sets  $I$  satisfies **Solecki's Property (\*)** if for any sequence  $K_n \in I, \forall n$ , there is a  $G_\delta$  set  $G$  such that  $\bigcup_n K_n \subseteq G$  and  $K(G) = \{K \in K(X) : K \subseteq G\} \subseteq I$ ; see [[Sol11](#)].

**Proposition 3.10.5.** *If  $X, E$  are as in [Proposition 3.10.1](#), then  $I_E$  satisfies Solecki's Property (\*).*

*Proof.* Let  $K_n \in I_E, \forall n$ . Then there is  $x \in X$  such that  $[x]_E \cap [\bigcup_n K_n]_E = \emptyset$  and thus if  $G = X \setminus [x]_E$ ,  $G$  is  $G_\delta$  and  $K(G) \subseteq I_E$ .  $\square$

In particular  $I_E$  admits a representation as in [[Sol11](#), Theorem 3.1].

A  $\sigma$ -ideal  $I$  of compact sets is **ccc** if there is no uncountable collection of pairwise disjoint compact sets which are not in  $I$ . Since for any CBER  $E$  every  $K \notin I_E$  is a complete section, it follows that  $I_E$  is ccc.

On the other hand, let  $I_E^*$  be the  $\sigma$ -ideal of subsets of  $X$  generated by  $I_E$ , i.e, for  $A \subset X, A \in I_E^* \iff \exists(K_n)(K_n \in I_E, \forall n, \text{ and } A \subseteq \bigcup_n K_n)$ . Then  $I_E^*$  is not ccc, in fact we have the following:

**Proposition 3.10.6.** *Let  $X, E$  be as in [Proposition 3.10.1](#) and moreover for every nonempty open set  $U \subseteq X$  there is a meager complete section contained in  $U$ . Then there is a homeomorphic embedding  $f: 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightarrow X$  such that for every  $\alpha \in 2^{\mathbb{N}}$ , we have  $f(\{\alpha\} \times \mathbb{N}^{\mathbb{N}}) \notin I_E^*$ .*

*Proof.* By [[KS95](#), Section 3, Lemma 9], it is enough to show that for every nonempty open  $U \subseteq X$ , there is a nowhere dense compact set  $K \subseteq U$  with  $K \notin I_E$ . This follows as in the proof of [Corollary 3.10.2](#).  $\square$

A  $\sigma$ -ideal  $I$  of compact sets has the **covering property** if for every  $\Sigma_1^1$  set  $A \subseteq X$ , either  $A \subseteq \bigcup_n K_n$ , where  $K_n \in I, \forall n$ , or else  $K(A) \subseteq I$ . It is **calibrated** if whenever  $K \in K(X)$  and  $K_n \subseteq K$  are such that  $K_n \in I, \forall n$ , and  $K(K \setminus \bigcup_n K_n) \subseteq I$ , then  $K \in I$ .

**Proposition 3.10.7.** *Let  $X, E$  be as in Proposition 3.10.1. Then  $I_E$  does not have the covering property and is not calibrated.*

*Proof.* Fix  $x \in X$  and let  $G = X \setminus [x]_E$ . This provides a counterexample to both properties.  $\square$

We next provide an example of a pair  $X, E$  satisfying all the properties of Proposition 3.10.6, and which therefore satisfies all the preceding propositions. We take  $X$  to be the collection of all subsets  $A$  of  $\mathbb{N}$  such that  $0 \in A, 1 \notin A$ , with the usual topology. We let then  $E$  be the restriction of many-one equivalence to  $X$ . It is easy to see that  $E$  is a  $K_\sigma$  CBER and every  $E$ -class is dense. Finally if  $U$  is an open subset of  $X$ , which we can assume that it has the form  $U = \{A \in X : F_1 \subseteq A, F_2 \cap A = \emptyset\}$ , for two disjoint finite subsets  $F_1, F_2$  of  $\mathbb{N}$ , then for a large enough number  $n$  the set  $K = \{A \in U : A \text{ contains only even numbers } > n\}$  is a meager complete section contained in  $U$ .

## 4 Generators and 2-adequate groups

For each infinite countable group  $\Gamma$  and standard Borel space  $X$  consider the shift action of  $\Gamma$  on  $X^\Gamma$  and let  $E(\Gamma, X)$  be the associated equivalence relation and  $E^{ap}(\Gamma, X)$  be its aperiodic part, i.e., the restriction of  $E(\Gamma, X)$  to the set of points with infinite orbits. Consider now a Borel action of  $\Gamma$  on an uncountable standard Borel space, which we can assume is equal to  $\mathbb{R}$ . Then the map  $f: X \rightarrow \mathbb{R}^\Gamma$  given by  $x \mapsto p_x$ , where  $p_x(\gamma) = \gamma^{-1} \cdot x$ , is an equivariant Borel embedding of this action to the shift action on  $\mathbb{R}^\Gamma$ . In particular for every aperiodic equivalence relation  $E$  induced by a Borel action of  $\Gamma$  we have that  $E \sqsubseteq_B^i E(\Gamma, \mathbb{R})$ , where for equivalence relations  $R, S$  on standard Borel spaces  $Y, Z$ , resp., we let  $R \sqsubseteq_B^i S$  iff there is an injective Borel reduction  $f: Y \rightarrow Z$  of  $R$  to  $S$  such that  $f(Y)$  is  $S$ -invariant. Thus every aperiodic equivalence relation  $E$  induced by a Borel action of  $\Gamma$  can be realized as (i.e., is Borel isomorphic to) the restriction of  $E^{ap}(\Gamma, \mathbb{R})$  to an invariant Borel set.

Now recall that for a Borel action of  $\Gamma$  on a standard Borel space  $X$  and  $n \in \{2, 3, \dots, \dots, \mathbb{N}\}$  an  **$n$ -generator** is a Borel partition  $X = \bigsqcup_{i < n} X_i$  such that  $\{\gamma \cdot X_i : \gamma \in \Gamma, i < n\}$  generates the Borel sets in  $X$ . This is equivalent to having a Borel equivariant embedding of the action to the shift action on  $n^\Gamma$ .

It is shown in [JKL02] that for every such action with infinite orbits there exists an  $\mathbb{N}$ -generator. It follows that every aperiodic equivalence relation  $E$  induced by a Borel action of  $\Gamma$  can be realized as the restriction of  $E^{ap}(\Gamma, \mathbb{N})$  to an invariant Borel set. In particular  $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, \mathbb{N})$ . However because of entropy considerations, even for the group  $\Gamma = \mathbb{Z}$ , it is not the case that every such action with invariant measure has a finite generator.

Weiss [Wei89] asked whether for  $\Gamma = \mathbb{Z}$  any Borel action without invariant measure admits a finite generator. Tserunyan [Tse15] showed that answer is affirmative for *any* infinite countable group  $\Gamma$  if the action is Borel isomorphic to a continuous action on a  $\sigma$ -compact Polish space. Then Hochman [Hoc19] provided a positive answer to Weiss' question (for  $\mathbb{Z}$ ). Finally this work culminated in the following complete answer:

**Theorem 4.0.1** (Hochman-Seward). *Every Borel action of a countable group on a standard Borel space without invariant measure admits a 2-generator.*

This however leaves open the possibility that every aperiodic CBER  $E$  induced by a Borel action of  $\Gamma$  can be realized as the restriction of  $E^{ap}(\Gamma, 2)$  to an invariant Borel set. This is clearly equivalent to the statement that  $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2)$  and it also equivalent to the statement that there is a Borel action of  $\Gamma$  that generates  $E$  and has a 2-generator. This leads to the following concept.

**Definition 4.0.2.** An infinite countable group  $\Gamma$  is called **2-adequate** if

$$E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2).$$

**Remark 4.0.3.** Thomas [Tho12] studies the question of when  $E(\Gamma, \mathbb{R}) \sim_B E(\Gamma, 2)$ .

The first result here is the following:

**Theorem 4.0.4.** *Every infinite countable amenable group is 2-adequate.*

*Proof.* Let  $X = \mathbb{R}^\Gamma, Y = 2^\Gamma, E = E^{ap}(\Gamma, \mathbb{R}), F = E^{ap}(\Gamma, 2)$ . Note that  $|\text{EINV}_F| = |\text{EINV}_E| = 2^{\aleph_0}$ , so fix a Borel bijection  $\pi: \text{EINV}_E \rightarrow \text{EINV}_F$ . Fix also the ergodic decompositions  $\{X_e\}_{e \in \text{EINV}_E}$  of  $E$  and  $\{Y_f\}_{f \in \text{EINV}_F}$  of  $F$ , resp. By the Ornstein-Weiss Theorem, see. e.g., [Kec21b, 7.25], let  $Z_e$  be an  $E$ -invariant Borel subset of  $X_e$  such that  $E|_{Z_e}$  is hyperfinite with unique invariant measure  $e$ . Again the construction of  $Z_e$  is effective enough, so that  $Z = \bigcup_e Z_e$  is Borel. Put  $X' = X \setminus Z$ , so that  $E|_{X'}$  is compressible.

Then, by Theorem 4.0.1, there is a Borel  $F$ -invariant subset  $Y'$  of  $Y$  such that  $E|_{X'} \cong_B F|_{Y'}$ , say by the Borel isomorphism  $g: X' \rightarrow Y'$ . Put  $W' = Y \setminus Y'$ .



Then let  $W_f$  be an  $F$ -invariant Borel subset of  $Y_f$  such that  $W_f \subseteq W'$  and  $F|W_f$  is hyperfinite with unique invariant measure  $f$ . Again the construction of  $W_f$  is effective enough, so that  $W = \bigcup_f W_f$  is Borel and there is a Borel isomorphism  $h_e$  of  $E|Z_e$  with  $F|W_{\pi(e)}$  such that moreover  $h = \bigcup_e h_e$  is Borel and thus a Borel isomorphism of  $E|Z$  with  $F|W$ . Then  $g \cup h$  shows that  $E \sqsubseteq_B^i F$  and the proof is complete.  $\square$

Thomas [Tho12, Page 391] asked the question of whether there is an infinite amenable  $\Gamma$  such that  $E(\Gamma, \mathbb{R}) \not\sqsubseteq_B E(\Gamma, 2)$ . [Theorem 4.0.4](#) provides a negative answer in a strong form.

To discuss other examples of 2-adequate groups, we will need the following strengthening of [Theorem 2.2.2](#).

**Proposition 4.0.5.** *Let  $E \in \mathcal{AE}$  and let  $R \subseteq E$  be hyperfinite. Then there is  $R \subseteq F \subseteq E$  with  $F \in \mathcal{AH}$ .*

*Proof.* Suppose  $E$  lives on the standard Borel space  $X$  and let

$$Y = \{x: [x]_E \text{ contains a finite nonempty set of finite } R\text{-classes}\}.$$

Then  $Y$  is  $E$ -invariant and  $E|Y$  is smooth, thus we can let  $F = E$  on  $Y$ . Let  $W = \{x: [x]_E \text{ contains no finite } R\text{-classes}\}$ . Then we can take  $F = R$  on  $W$ .

So we can assume that each  $E$ -class contains infinitely many finite  $R$ -classes. Let  $Z = \{x: [x]_R \text{ is finite}\}$ . Then  $R|Z$  is  $R$ -invariant and smooth, so let  $S$  be a Borel selector and  $T$  the associated Borel transversal  $T = \{x: S(x) = x\}$ . Then, by [Theorem 2.2.2](#), let  $F'$  be a hyperfinite aperiodic Borel equivalence relation on  $T$  such that  $F' \subseteq E|T$ . Let then  $F''$  be the equivalence relation on  $Z$  defined by  $xF''y \iff S(x)F'S(y)$ . It is clearly aperiodic, hyperfinite, and  $R|Z \subseteq F'' \subseteq E|Z$ . Finally put  $F = F'' \cup R|(X \setminus Z)$ .  $\square$

We also consider the following class of countable groups.

**Definition 4.0.6.** A countable group  $\Gamma$  is **hyperfinite generating** if for every  $E \in \mathcal{AH}$  there is a Borel action of  $\Gamma$  that generates  $E$ .

We now have the next result that generalizes [Proposition 4.0.5](#) from  $\mathbb{Z}$  to any hyperfinite generating group. The proof is similar, noting that any smooth aperiodic CBER can be generated by a Borel action of any infinite countable group.

**Proposition 4.0.7.** *Let  $E \in \mathcal{AE}$  and let  $R \subseteq E$  be generated by a Borel action of  $\Gamma$ , where  $\Gamma$  is a hyperfinite generating group. Then there is  $R \subseteq F \subseteq E$  with  $F \in \mathcal{AE}$  generated by a Borel action of  $\Gamma$ .*

**Proposition 4.0.8.** *Let  $\Gamma$  be any countable group and  $\Delta$  a hyperfinite generating, 2-adequate group. Then  $\Gamma \star \Delta$  is 2-adequate.*

*Proof.* Fix a Borel action  $\mathbf{a}$  of  $\Gamma \star \Delta$  on an uncountable standard Borel space  $X$  generating an aperiodic equivalence relation that we denote by  $E_{\mathbf{a}}$ . Let  $\mathbf{b} = \mathbf{a}|_{\Delta}$ ,  $\mathbf{c} = \mathbf{a}|_{\Gamma}$  and denote by  $E_{\mathbf{b}}, E_{\mathbf{c}}$  the associated equivalence relations, so that  $E_{\mathbf{a}} = E_{\mathbf{b}} \vee E_{\mathbf{c}}$ . By [Proposition 4.0.7](#) find a Borel action  $\mathbf{b}'$  of  $\Delta$  such that  $E_{\mathbf{b}'}$  is aperiodic and  $E_{\mathbf{b}} \subseteq E_{\mathbf{b}'} \subseteq E_{\mathbf{a}}$ , so that  $E_{\mathbf{a}} = E_{\mathbf{b}'} \vee E_{\mathbf{c}}$ . Let now  $\mathbf{a}'$  be the action of  $\Gamma \star \Delta$  such that  $\mathbf{a}'|_{\Delta} = \mathbf{b}'$ ,  $\mathbf{a}'|_{\Gamma} = \mathbf{c}$ , so that  $E_{\mathbf{a}'} = E_{\mathbf{a}}$ . Since  $\mathbf{b}'$  has a 2-generator, so does  $\mathbf{a}'$  and the proof is complete.  $\square$

It will be shown in [Corollary 5.1.2](#) that all groups that have an infinite amenable factor are hyperfinite generating. Thus we have:

**Corollary 4.0.9.** *The free product of any countable group with a group that has an infinite amenable factor and thus, in particular, the free groups  $\mathbb{F}_n$ ,  $1 \leq n \leq \infty$ , are 2-adequate.*

The following is immediate:

**Proposition 4.0.10.** *If  $\Gamma, \Delta$  are countable groups, every aperiodic equivalence relation induced by a Borel action of  $\Gamma$  can be also induced by a Borel action of  $\Delta$ ,  $\Delta$  is a factor of  $\Gamma$  and  $\Delta$  is 2-adequate, so is  $\Gamma$ . In particular, for any  $1 \leq n \leq \infty$ , every  $n$ -generated countable group that factors onto  $\mathbb{F}_n$  is 2-adequate.*

The next two results owe a lot to some crucial observations by Brandon Seward.

**Proposition 4.0.11.** *Let  $\Gamma$  be  $n$ -generated,  $1 \leq n \leq \infty$ . Then  $\Gamma \times \mathbb{F}_n$  is 2-adequate. In particular, all products  $\mathbb{F}_m \times \mathbb{F}_n$ ,  $1 \leq m, n \leq \infty$ , are 2-adequate.*

*Proof.* Let  $\{\gamma_i\}_{i < n}$  be generators for  $\Gamma$  and let  $\{\alpha_i\}_{i < n}$  be free generators for  $\mathbb{F}_n$ . Consider a Borel action  $\mathbf{a}$  of  $\Gamma \times \mathbb{F}_n$  with  $E_{\mathbf{a}}$  aperiodic. Then the equivalence relation  $E_i$  generated by  $\mathbf{a}|_{\langle \gamma_i, \alpha_i \rangle}$  is generated by a Borel action of  $\mathbb{Z}^2$  thus is hyperfinite, see, e.g., [[Kec21b](#), 7.F], and so is given by a Borel action  $\mathbf{a}_i$  of  $\mathbb{Z}$ . Let  $\mathbf{b}$  the Borel action of  $\mathbb{F}_n$  in which the generator  $\alpha_i$  acts like  $\mathbf{a}_i$ . Then  $E_{\mathbf{b}} = \bigvee_i E_i = E$  and the proof is complete by [Proposition 4.0.10](#).  $\square$

Finally not every infinite countable group is 2-amenable. The argument below follows the pattern of the proofs in [[Tho12](#), Section 6].

**Theorem 4.0.12.** *The group  $\mathrm{SL}_3(\mathbb{Z})$  is not 2-adequate.*

*Proof.* Assume that  $\Gamma = \mathrm{SL}_3(\mathbb{Z})$  is 2-adequate, towards a contradiction. Then in particular  $E^{ap}(\Gamma, 3) \cong_B E^{ap}(\Gamma, 2)$ , say via the Borel isomorphism  $f$ . Let  $\mu$  be the usual product of the uniform measure on  $3^\Gamma$ . Then  $\nu = f_*\mu$  is an ergodic, invariant measure for the shift action of  $\Gamma$  on  $2^\Gamma$ , thus by Stuck-Zimmer [SZ94] this shift action is free  $\nu$ -a.e. This gives a contradiction by the arguments in [Tho12, Section 6].  $\square$

We conclude this section with the following problem:

**Problem 4.0.13.** *Characterize the 2-adequate groups.*

## 5 Additional results

### 5.1 Hyperfinite generating groups

We introduced in Section 4 the concept of hyperfinite generating groups. We will establish here some equivalent formulations of this concept and in particular prove the fact mentioned in the paragraph after Proposition 4.0.8. Below we let  $\mu$  be the product of the uniform measure on  $2^\mathbb{N}$  and by  $[E_0] \leq \mathrm{Aut}(2^\mathbb{N}, \mu)$  the usual measure theoretic full group of the pmp equivalence relation  $E_0$ . For a countable group  $\Delta \leq [E_0]$ , we denote by  $E_\Delta$  the subequivalence relation of  $E_0$  induced by the action of  $\Delta$  on  $2^\mathbb{N}$ . This is again understood to be defined only  $\mu$ -a.e.

Below an **IRS** on a countable group  $\Gamma$  is a measure on the space of subgroups of  $\Gamma$  invariant under conjugation. We say that an IRS  $\mu$  has some property P if  $\mu$ -almost all  $\Delta \leq \Gamma$  have property P. Finally a subgroup  $\Delta \leq \Gamma$  is **co-amenable** if the action of  $\Gamma$  on  $\Gamma/\Delta$  is amenable, i.e., admits a finitely additive probability measure.

**Proposition 5.1.1.** *Let  $\Gamma$  be an infinite countable group. Then the following are equivalent:*

- (i)  $\Gamma$  is hyperfinite generating;
- (ii) There is a Borel action of  $\Gamma$  that generates  $E_0$ ;
- (iii)  $\Gamma$  admits a Borel action which generates a non-compressible, aperiodic hyperfinite equivalence relation;
- (iv)  $\Gamma$  admits a factor  $\Delta \leq [E_0]$  such that  $E_\Delta$  has a  $\mu$ -positive set of infinite orbits.

Moreover, if  $\Gamma$  is hyperfinite generating,  $\Gamma$  admits a co-amenable IRS with infinite index.

*Proof.* Clearly (i)  $\implies$  (ii)  $\implies$  (iii). We next prove that (iii)  $\implies$  (iv). Indeed (iii) implies that there is a Borel action of  $\Gamma$  on a standard Borel space  $X$  generating

an aperiodic equivalence relation  $E$  that has an ergodic, invariant measure  $\mu$ . This action induces a homomorphism  $\pi: \Gamma \rightarrow [E]$ , the measure theoretic full group of  $E$ , with respect to  $\mu$ . If  $\Delta = \pi(\Gamma) \leq [E]$ , then  $\Delta$  generates  $E$  (again  $\mu$ -a.e). But by Ornstein-Weiss and Dye, see, e.g., [Kec21b, 7.8 and 7.25],  $E$  and  $E_0$  are measure theoretically isomorphic, which proves (iv).

We now show that (iv)  $\implies$  (i). Fix  $E \in \mathcal{AH}$  which lives on a space  $X$ . If  $E$  is compressible, then it is generated by a Borel action of  $\Gamma$ , by [DJK94, 11.2]. Otherwise consider the ergodic decomposition  $\{X_e\}_{e \in \text{EINV}_E}$  of  $E$ . Now (iv) implies (iii) and it follows that  $\Gamma$  has a Borel action on a standard Borel space  $Z$  which generates an aperiodic hyperfinite equivalence relation  $F$ , which has an ergodic, invariant measure  $\mu$ . Find then, using Dye's Theorem, see, e.g., [Kec21b, 7.8], invariant Borel sets  $Y_e \subseteq X_e$  with  $e(Y_e) = 1$  and  $Z_e \subseteq Z$  with  $\mu(Z_e) = 1$  such that  $E|_{Y_e}$  and  $F|_{Z_e}$  are Borel isomorphic. Then  $E|_{Y_e}$  can be generated by a Borel action of  $\Gamma$ , and, by the effectivity of this construction, we also have that  $Y = \bigcup_e Y_e$  is Borel and putting together the action of  $\Gamma$  on each  $Y_e$ , we get a Borel action of  $\Gamma$  on  $Y$  which generates  $E|_Y$ . Since  $E|(X \setminus Y)$  is compressible, this shows that  $E$  is generated by a Borel action of  $\Gamma$ .

Finally the last statement follows as in the proof of (vii)  $\implies$  (x) in the last paragraph of [BK20, Appendix D] (finite generation is not required there).  $\square$

**Corollary 5.1.2.** *Every countable group that has an infinite amenable factor is hyperfinite generating.*

*Proof.* If  $\Gamma$  is infinite amenable, consider its shift action on  $2^\Gamma$ , equipped with the product of the uniform measure, with associated equivalence relation  $E = E(\Gamma, 2)$ . Then  $E$  and  $E_0$  are measure theoretically isomorphic, so the measure theoretic full group of  $E$  is isomorphic to  $[E_0]$ . Since  $\Gamma \leq [E]$  we have an embedding  $\pi: \Gamma \rightarrow [E_0]$  such that if  $\Delta = \pi(\Gamma)$ , then  $E_\Delta = E_0$ , which completes the proof.  $\square$

It also immediately follows from [Mil06, Theorem 13] that every countable group that has a factor of the form  $\Gamma \star \Delta$ , where  $\Gamma, \Delta$  are non-trivial subgroups of  $[E_0]$ , is hyperfinite generating.

On the other hand, not every infinite countable group is hyperfinite generating.

**Proposition 5.1.3.** *No infinite countable group with property (T) is hyperfinite generating.*

*Proof.* See, for example, the proof of [Kec10, Proposition 4.14].  $\square$

It is also shown in [Kec10, page 29] that there are groups that do not have property (T) and are not hyperfinite generating.

The following is an open problem.

**Problem 5.1.4.** *Characterize the hyperfinite generating groups.*

## 5.2 Dynamically compressible groups

In the course of the previous investigations the following property of countable groups came up. As usual we employ the notation  $E_\Gamma^X$  for the equivalence relation induced by a Borel action of a countable group  $\Gamma$  on a standard Borel space  $X$ .

**Definition 5.2.1.** An infinite countable group  $\Gamma$  is called **dynamically compressible** if for every aperiodic  $E_\Gamma^X$ , there is a compressible  $E_\Gamma^Y$  with  $E_\Gamma^X \leq_B E_\Gamma^Y$ .

Here is an equivalent formulation of this notion.

**Proposition 5.2.2.** *A countable group  $\Gamma$  is dynamically compressible iff for every aperiodic  $E_\Gamma^X$ ,  $E_\Gamma^X \times I_{\mathbb{N}}$  is induced by a Borel action of  $\Gamma$*

*Proof.* Since  $E_\Gamma^X \times I_{\mathbb{N}} \leq_B E_\Gamma^X$ , if  $E_\Gamma^X \leq_B E_\Gamma^Y$ , with  $E_\Gamma^Y$  compressible, then  $E_\Gamma^X \times I_{\mathbb{N}} \leq_B E_\Gamma^Y$ , therefore  $E_\Gamma^X \times I_{\mathbb{N}} \sqsubseteq_B E_\Gamma^Y$  by [Kec21b, 2.27].  $\square$

We now have:

**Proposition 5.2.3.** *Every infinite countable amenable group is dynamically compressible.*

*Proof.* Consider any aperiodic  $E = E_\Gamma^X$ , which we can clearly assume is not compressible, so admits an invariant measure. Then let  $\{X_e\}_{e \in \text{EINV}_E}$  be its ergodic decomposition. Then there is a Borel set  $Y_e \subseteq X_e$  with  $e(Y_e) = 1$  such that  $E|_{Y_e}$  is hyperfinite, thus  $E|_{Y_e} \leq_B E_t$ . As usual  $Y = \bigcup_e Y_e$  is Borel and  $E|_Y \leq_B \mathbb{R}E_t \leq_B E_t$ . Now  $E|(X \setminus Y)$  is compressible and  $E_t$  is induced by a Borel action of  $\Gamma$  by [DJK94, 11.2], so the proof is complete.  $\square$

**Proposition 5.2.4.** *If  $\mathbb{F}_2 \leq \Gamma$ , then  $\Gamma$  is dynamically compressible.*

*Proof.* Let  $E_\Gamma^X$  be aperiodic. Then  $E_\Gamma^X = E_{\mathbb{F}_\infty}^X \leq_B E_{\mathbb{F}_\infty}^X \times I_{\mathbb{N}} = E_{\mathbb{F}_\infty}^Y$ , for  $Y = X \times \mathbb{N}$ . Now  $\mathbb{F}_\infty \leq \Gamma$ , so by using the inducing construction from the action of  $\mathbb{F}_\infty$  on  $Y$ , see [BK96, 2.3.5], we have  $E_{\mathbb{F}_\infty}^Y \leq_B E_Z^\Gamma$  for some compressible  $E_Z^\Gamma$ .  $\square$

Therefore only the groups that are not amenable but do not contain  $\mathbb{F}_2$  can possibly fail to be dynamically compressible. But even among those there exist dynamically compressible groups.

**Proposition 5.2.5.** *Let  $\Gamma$  be a countable group for which there is an infinite group  $\Delta$  such that  $\Gamma \times \Delta \leq \Gamma$ . Then  $\Gamma$  is dynamically compressible.*

*Proof.* Let  $E_\Gamma^X$  be aperiodic. Then for  $Y = X \times \mathbb{N}$ ,  $E_\Gamma^X \leq_B E_\Gamma^X \times I_\mathbb{N} = E_{\Gamma \times \Delta}^Y \leq_B E_\Gamma^Z$ , where  $E_\Gamma^Z$  is obtained by inducing from the action of  $\Gamma \times \Delta$  on  $Y$ .  $\square$

As a result any countable group of the form  $\Gamma \times \Delta^{<\mathbb{N}}$ , for an infinite  $\Delta$ , is dynamically compressible. Take now  $\Gamma$  to be any group that is not amenable and does not contain  $\mathbb{F}_2$  and consider  $G = \Gamma \times \mathbb{Z}^{<\mathbb{N}}$ . Then  $G$  is dynamically compressible and clearly is not amenable. Moreover it does not contain  $\mathbb{F}_2$  because of the following standard fact.

**Proposition 5.2.6.** *Let  $G, H$  be two groups such that  $\mathbb{F}_2 \leq G \times H$ . Then  $\mathbb{F}_2 \leq G$  or  $\mathbb{F}_2 \leq H$ .*

*Proof.* Let  $\pi: \mathbb{F}_2 \rightarrow H$  be the second projection, If it has trivial kernel, then  $\mathbb{F}_2 \leq H$ . Else either  $\mathbb{F}_2 \leq \ker(\pi) \leq G$  or  $\ker(\pi) \cong \mathbb{Z}$ . In the latter case, by [LS01, 3.110],  $[\mathbb{F}_2 : \ker(\pi)]$  is finite, so by [LS01, 3.9],

$$[\mathbb{F}_2 : \ker(\pi)] = \frac{\text{rank}(\ker(\pi)) - 1}{\text{rank}(\mathbb{F}_2) - 1} = 0,$$

a contradiction.  $\square$

We now have the following open problem:

**Problem 5.2.7.** *Is every infinite countable group dynamically compressible?*

We note that  $\Gamma$  fails to be dynamically compressible iff there is some aperiodic  $E_\Gamma^X$  such that every  $E_\Gamma^X \leq_B E_\Gamma^Y$  admits an invariant measure.

We conclude with the following interesting consequence of Proposition 5.2.4. Let  $\Gamma = \text{SL}_3(\mathbb{Z})$  and consider the shift action of  $\Gamma$  on  $\mathbb{R}^\Gamma$  and denote by  $E = F(\Gamma, \mathbb{R})$  the restriction of  $E(\Gamma, \mathbb{R})$  to the free part of the action. Then, by Proposition 5.2.4,  $E \times I_\mathbb{N}$  is induced by a Borel action of  $\Gamma$ . On the other hand,  $E \times I_\mathbb{N}$  cannot be induced by a free Borel action of  $\Gamma$ , since if that was the case then  $E \times I_\mathbb{N} \sqsubseteq_B^i E$ , contradicting the Addendum following [CK18, 5.28].

## 6 Open problems

For the convenience of the reader, we collect here some of the main open problems discussed in this paper.

**Problem 6.0.1.** (*Problem 2.3.2*) *Let  $\kappa \geq 1$ . Does  $\mathcal{AT}_\kappa$  have a  $\subseteq_B$ -maximum element?*

**Problem 6.0.2.** (*Problem 3.2.10*) Does every non-smooth  $E \in \mathcal{AE}$  have any of the topological realizations stated in *Definition 3.2.2*? In particular, does every non-smooth  $E \in \mathcal{AE}$  admit a compact action realization?

**Problem 6.0.3.** (*Problem 3.2.11*) If a CBER admits a compact action realization, does it admit one in which the underlying space is  $2^{\mathbb{N}}$ ?

**Problem 6.0.4.** (*Problem 3.2.12*) Is every non-smooth  $E \in \mathcal{AE}$  Borel bireducible to some  $F \in \mathcal{AE}$  which has any of the topological realizations stated in *Definition 3.2.2*? In particular, can one find such an  $F$  that admits a compact action realization?

Note that by *Theorem 3.3.11*, every non-smooth  $E \in \mathcal{AE}$  is Borel bireducible to some  $F \in \mathcal{AE}$  which has a compact action realization iff every non-smooth compressible  $E \in \mathcal{AE}$  has a compact action realization.

**Problem 6.0.5.** (*Problem 3.3.3*) Is it true that for an arbitrary infinite countable group  $\Gamma$ ,  $E^{ap}(\Gamma, 2^{\mathbb{N}})$  admits a compact action realization?

**Problem 6.0.6.** (*Problem 3.3.6*) Let  $E \in \mathcal{AE}$  be on a standard Borel space  $X$  and let  $\mu$  be a measure on  $X$  such that the restriction of  $E$  to any invariant Borel set of measure 1 is not smooth. Is there is an invariant Borel set  $Y \subseteq X$  with  $\mu(Y) = 1$  such that  $E|_Y$  admits a compact action realization?

**Problem 6.0.7.** (*Problem 3.3.15*) Does an arbitrary (not necessarily compressible) aperiodic, universal CBER admit a compact action realization?

**Problem 6.0.8.** (*Problem 3.4.2*) Does Turing equivalence  $\equiv_T$  on  $2^{\mathbb{N}}$  admit a compact action realization?

**Problem 6.0.9.** (*Problem 3.4.11*) Is there a Baire class 1 map that is an isomorphism between  $\equiv_T$  and an equivalence relation given by a continuous group action on  $\mathbb{N}^{\mathbb{N}}$ ?

**Problem 6.0.10.** (*Problem 3.4.12*) Is there a Borel map  $\Phi : 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  that is an isomorphism between  $\equiv_T$  and an equivalence relation given by a continuous group action on  $\mathbb{N}^{\mathbb{N}}$  such that  $\Phi(x) \equiv_T x'$  on a cone?

**Problem 6.0.11.** (*Problem 3.5.7*) In *Theorem 3.5.3*, can one replace (iii) by “ $\mathbf{a}$  is finitely compressible” and similarly for (iv).

**Problem 6.0.12.** (*Problem 3.6.7*) Does *Corollary 3.6.6* hold with  $\mathbb{F}_2$  instead of  $\mathbb{F}_3, \mathbb{F}_4$ ?

**Problem 6.0.13.** (*Problem 3.6.8*) Does every non-smooth aperiodic CBER have a realization as a subshift of  $2^\Gamma$  for some group  $\Gamma$ ? Also does it have a realization as a minimal subshift?

**Problem 6.0.14.** (*Problem 3.6.12*) Is  $\mathbb{F}_2$  minimal subshift universal? More generally, is every group that contains  $\mathbb{F}_2$  minimal subshift universal?

**Problem 6.0.15.** (*Problem 3.7.4*) Is it true that a group  $\Gamma$  contains  $\mathbb{F}_2$  iff there is a compressible, orbit-universal subshift of  $2^\Gamma$ ?

**Problem 6.0.16.** (*Problem 3.8.1*) Let  $\Gamma$  be an infinite group. Is  $\text{Sh}_{\text{hyp}}(\Gamma, \mathbb{I}^\mathbb{N})$  comeager in  $\text{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$ ?

**Problem 6.0.17.** (*Problem 3.8.2*) Let  $\Gamma$  be an infinite group. What is the exact descriptive complexity of  $\text{Sh}_{\text{hyp}}(\Gamma, \mathbb{I}^\mathbb{N})$  in  $\text{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$ ?

**Problem 6.0.18.** (*Problem 3.8.16*) If  $\Gamma$  is finitely generated and amenable, is it true that for comeager many  $\mathbf{a} \in \text{Act}(\Gamma, 2^\mathbb{N})$ ,  $\text{INV}(\mathbf{a})$  is a Bauer simplex, i.e.,  $\text{EINV}(\mathbf{a})$  is closed in  $\text{INV}(\mathbf{a})$ ?

**Problem 6.0.19.** (*Problem 3.8.17*) Let  $\mathbf{a} \in \text{Act}(\mathbb{F}_2, 2^\mathbb{N})$  be the generic action of  $\mathbb{F}_2$  on  $2^\mathbb{N}$ . Is the equivalence relation  $E_{\mathbf{a}}$  hyperfinite?

**Problem 6.0.20.** (*Problem 3.9.3*) Let  $E$  be an aperiodic CBER. Does  $E$  have a minimal  $F_\sigma$  realization? If  $E$  is non-smooth, does  $E$  have a minimal  $K_\sigma$  realization?

**Problem 6.0.21.** (*Problem 3.9.6*) Does every aperiodic CBER admit a compactly generated realization?

**Problem 6.0.22.** (*Problem 4.0.13*) Characterize the 2-adequate groups.

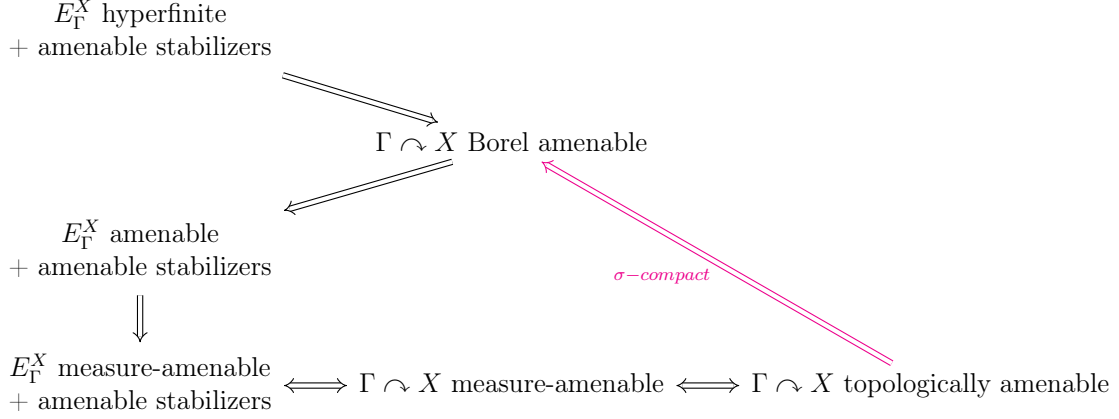
**Problem 6.0.23.** (*Problem 5.1.4*) Characterize the hyperfinite generating groups.

**Problem 6.0.24.** (*Problem 5.2.7*) Is every infinite countable group dynamically compressible?



# A Amenable actions

The purpose of this appendix is to explain the following implications for a continuous action  $\Gamma \curvearrowright X$  of a countable group on a Polish space. Recall that  $E_\Gamma^X$  is the induced orbit equivalence relation and all the concepts in the diagram below are defined in Section 3.8, (B), except for topological amenability which is defined below in Appendix A.3.



## A.1 Borel amenability

We first have the following result:

**Theorem A.1.1.** *Let  $\Gamma \curvearrowright X$  be a Borel action of a countable group on a standard Borel space, and consider the following statements:*

- (1)  $E_\Gamma^X$  is hyperfinite and every stabilizer is amenable.
- (2)  $\Gamma \curvearrowright X$  is Borel amenable.
- (3)  $E_\Gamma^X$  is amenable and every stabilizer is amenable.

Then (1)  $\implies$  (2)  $\implies$  (3).

*Proof.* Let  $E := E_\Gamma^X$ .

- (1)  $\implies$  (2): Since  $E$  is hyperfinite, it is amenable in a strong sense: there is a sequence  $p_n : E \rightarrow [0, 1]$  of Borel functions, such that  $p_n^x$  is a probability measure supported on  $[x]_E$ , for every  $(x, y) \in E$ , we have  $\|p_n^x - p_n^y\|_1 \rightarrow 0$ , and additionally, for every  $y$ , there are only finitely many  $x$  with  $p_n^x(y) > 0$ .

Let  $\alpha: E \rightarrow \Gamma$  be a Borel function such that for every  $(x, y) \in E$ , we have  $y = \alpha_x^y \cdot x$  and  $\alpha_y^x \alpha_x^y = 1$ . Write  $\Gamma = \bigcup_n S_n$  as an increasing union of finite subsets.

We claim that there is a sequence  $q_n: X \rightarrow \text{Prob}(\Gamma)$  of Borel functions with  $q_n^x$  supported on  $\Gamma_x$ , such that for every  $(x, y) \in E$  with  $p^x(y) > 0$  and every  $\gamma \in S_n$ , we have  $\|q_n^y - \alpha_{\gamma \cdot x}^y \gamma \alpha_y^x \cdot q_n^y\|_1 < \frac{1}{n}$ . To see this, for every  $y \in X$ , by amenability of  $\Gamma_y$ , let  $A_n^y$  be the least (in some enumeration) finite subset of  $\Gamma$  such that  $A_n^y \subseteq \Gamma_y$  and

$$\frac{|A_n^y \Delta A_n^y \alpha_x^y \gamma^{-1} \alpha_y^{\gamma \cdot x}|}{|A_n^y|} < \frac{1}{n}$$

for every  $x \in [y]_E$  with  $p^x(y) > 0$  and every  $\gamma \in S_n$ . Then let  $q_n^y := \frac{1}{|A_n^y|} \mathbf{1}_{A_n^y}$  be the uniform distribution on  $A_n^y$ . Then

$$\|q_n^y - \alpha_{\gamma \cdot x}^y \gamma \alpha_y^x \cdot q_n^y\|_1 = \frac{\|\mathbf{1}_{A_n^y} - \mathbf{1}_{A_n^y \alpha_x^y \gamma^{-1} \alpha_y^{\gamma \cdot x}}\|_1}{|A_n^y|} = \frac{|A_n^y \Delta A_n^y \alpha_x^y \gamma^{-1} \alpha_y^{\gamma \cdot x}|}{|A_n^y|} < \frac{1}{n}.$$

Let  $r_n: X \rightarrow \text{Prob}(\Gamma)$  be defined by

$$r_n^x(\gamma) = p_n^x(\gamma \cdot x) q_n^{\gamma \cdot x}(\gamma \alpha_{\gamma \cdot x}^x).$$

Let  $x \in X$  and  $\gamma \in \Gamma$ . Then

$$\begin{aligned} \|r_n^{\gamma \cdot x} - \gamma \cdot r_n^x\|_1 &= \sum_{\delta \in \Gamma} |r_n^{\gamma \cdot x}(\delta) - r_n^x(\delta \gamma)| \\ &= \sum_{\delta \in \Gamma} |p_n^{\gamma \cdot x}(\delta \gamma \cdot x) q_n^{\delta \gamma \cdot x}(\delta \alpha_{\delta \gamma \cdot x}^{\gamma \cdot x}) - p_n^x(\delta \gamma \cdot x) q_n^{\delta \gamma \cdot x}(\delta \gamma \alpha_{\delta \gamma \cdot x}^x)| \\ &= \sum_{y \in [x]_E} \sum_{\lambda \in \Gamma_y} |p_n^{\gamma \cdot x}(y) q_n^y(\lambda) - p_n^x(y) q_n^y(\lambda \alpha_{\gamma \cdot x}^y \gamma \alpha_y^x)| \\ &\leq \sum_{y \in [x]_E} |p_n^{\gamma \cdot x}(y) - p_n^x(y)| \sum_{\lambda \in \Gamma_y} q_n^y(\lambda) \\ &\quad + \sum_{y \in [x]_E} p_n^x(y) \sum_{\lambda \in \Gamma_y} |q_n^y(\lambda) - q_n^y(\lambda \alpha_{\gamma \cdot x}^y \gamma \alpha_y^x)|. \end{aligned}$$

The first term is equal to  $\|p_n^{\gamma \cdot x} - p_n^x\|_1$ , which converges to 0. If  $\gamma \in S_n$ , then the second term is equal to

$$\sum_{\substack{y \in [x]_E \\ p_n^x(y) > 0}} p_n^x(y) \|q_n^y - \alpha_{\gamma \cdot x}^y \gamma \alpha_y^x \cdot q_n^y\|_1 < \sum_{\substack{y \in [x]_E \\ p_n^x(y) > 0}} p_n^x(y) \frac{1}{n} = \frac{1}{n} \rightarrow 0,$$

so the whole expression converges to 0.

(2)  $\implies$  (3): Let  $p_n: X \rightarrow \text{Prob}(\Gamma)$  witness the Borel amenability of the action  $\Gamma \curvearrowright X$ .

To show that  $E$  is amenable, define  $q_n: E \rightarrow [0, 1]$  by

$$q_n^x(y) := \sum_{\substack{\gamma \in \Gamma \\ \gamma \cdot x = y}} p_n^x(\gamma).$$

Now if  $x \in X$  and  $\gamma \in \Gamma$ , then we have

$$\begin{aligned} \|q_n^{\gamma \cdot x} - q_n^x\|_1 &= \sum_{y \in [x]_E} \left| \sum_{\substack{\delta \in \Gamma \\ \delta \gamma \cdot x = y}} p_n^{\gamma \cdot x}(\delta) - \sum_{\substack{\lambda \in \Gamma \\ \lambda \cdot x = y}} p_n^x(\lambda) \right| \\ &= \sum_{y \in [x]_E} \left| \sum_{\substack{\delta \in \Gamma \\ \delta \gamma \cdot x = y}} p_n^{\gamma \cdot x}(\delta) - \sum_{\substack{\delta \in \Gamma \\ \delta \gamma \cdot x = y}} p_n^x(\delta \gamma) \right| \\ &\leq \sum_{y \in [x]_E} \sum_{\substack{\delta \in \Gamma \\ \delta \gamma \cdot x = y}} |p_n^{\gamma \cdot x}(\delta) - p_n^x(\delta \gamma)| \\ &= \sum_{\delta \in \Gamma} |p_n^{\gamma \cdot x}(\delta) - p_n^x(\delta \gamma)| \\ &= \|p_n^{\gamma \cdot x} - \gamma \cdot p_n^x\|_1 \\ &\rightarrow 0. \end{aligned}$$

Thus  $E$  is amenable.

Now let  $x \in X$ . To see that  $\Gamma_x$  is amenable, let  $T$  be a transversal for left cosets of  $\Gamma_x$  in  $\Gamma$ , and define  $q_n \in \text{Prob}(\Gamma_x)$  by

$$q_n(\gamma) := \sum_{t \in T} p_n^x(t\gamma).$$

Then for every  $\gamma \in \Gamma_x$ , we have

$$\begin{aligned}
\|q_n - \gamma \cdot q_n\|_1 &= \sum_{\delta \in \Gamma_x} |q_n(\delta) - q_n(\delta\gamma)| \\
&= \sum_{\delta \in \Gamma_x} \left| \sum_{t \in T} p_n^x(t\delta) - \sum_{t \in T} p_n^x(t\delta\gamma) \right| \\
&\leq \sum_{\delta \in \Gamma_x} \sum_{t \in T} |p_n^x(t\delta) - p_n^x(t\delta\gamma)| \\
&= \sum_{\lambda \in \Gamma} |p_n^x(\lambda) - p_n^x(\lambda\gamma)| \\
&= \|p_n^x - \gamma \cdot p_n^x\|_1 \\
&= \|p_n^{\gamma \cdot x} - \gamma \cdot p_n^x\|_1 \\
&\rightarrow 0.
\end{aligned}$$

Thus  $\Gamma_x$  is amenable. □

## A.2 Measure amenability

By [Theorem A.1.1](#) and the Connes-Feldman-Weiss theorem, see, e.g., [[KM04](#), Theorem 10.1], we have the following analogue of [[AEG94](#)] (see also [[ADR00](#), Corollary 5.3.33]):

**Theorem A.2.1.** *Let  $\Gamma \curvearrowright X$  be a Borel action of a countable group on a standard Borel space, and let  $\mu$  be a Borel probability measure on  $X$ . Then the following are equivalent:*

- (1)  $\Gamma \curvearrowright X$  is  $\mu$ -amenable.
- (2)  $E_\Gamma^X$  is  $\mu$ -amenable and  $\mu$ -a.e. stabilizer is amenable.

**Corollary A.2.2.** *Let  $\Gamma \curvearrowright X$  be a Borel action of a countable group on a standard Borel space. Then the following are equivalent:*

- (1)  $\Gamma \curvearrowright X$  is measure-amenable.
- (2)  $E_\Gamma^X$  is measure-amenable and every stabilizer is amenable.

### A.3 Topological amenability

Let  $\Gamma$  be a countable group, and let  $X$  be a Polish space. A continuous action  $\Gamma \curvearrowright X$  is **topologically amenable** if for every finite  $S \subseteq \Gamma$ , every compact  $K \subseteq X$ , and every  $\varepsilon > 0$ , there is some continuous  $p: X \rightarrow \text{Prob}(\Gamma)$  such that

$$\max_{\substack{\gamma \in S \\ x \in K}} \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1 < \varepsilon.$$

Topological amenability is equivalent to measure amenability (see [ADR00, Theorem 3.3.7] for the locally compact case, also the proof of [BO08, Proposition 5.2.1]):

**Theorem A.3.1.** *Let  $\Gamma \curvearrowright X$  be a continuous action of a countable group on a Polish space. Then the following are equivalent:*

- (1)  $\Gamma \curvearrowright X$  is topologically amenable.
- (2)  $\Gamma \curvearrowright X$  is measure-amenable.

Moreover, if  $X$  is  $\sigma$ -compact, then these are also equivalent to

- (3)  $\Gamma \curvearrowright X$  is Borel amenable.

The following lemma says that in the definition of  $\mu$ -amenability, we can upgrade the Borel functions to continuous ones:

**Lemma A.3.2.** *Let  $\Gamma \curvearrowright X$  be a continuous action of a countable group on a Polish space, and let  $\mu$  be a Borel probability measure on  $X$ . Then the following are equivalent:*

1.  $\Gamma \curvearrowright X$  is  $\mu$ -amenable.
2. For every finite  $S \subseteq \Gamma$  and every  $\varepsilon > 0$ , there is some continuous  $p: X \rightarrow \text{Prob}(\Gamma)$  such that for every  $\gamma \in S$ , we have

$$\int_X \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1 d\mu(x) < \varepsilon.$$

*Proof of Lemma A.3.2.* It suffices to show that for every Borel  $p: X \rightarrow \text{Prob}(\Gamma)$  and every  $\varepsilon > 0$ , there is some continuous  $q: X \rightarrow \text{Prob}(\Gamma)$  such that

$$\int_X \|p - q\|_1 d\mu < \varepsilon.$$

By Lusin's theorem [Kec95, 17.12], there is a closed  $F \subseteq X$  with  $\mu(F) > 1 - \frac{\varepsilon}{2}$  such that  $p|_F$  is continuous. By Dugundji's extension theorem [Dug51, 4.1], there is some continuous extension  $q: X \rightarrow \text{Prob}(\Gamma)$  of  $p|_F$ . Then  $p$  and  $q$  agree on  $F$ , so we are done.  $\square$

*Proof of Theorem A.3.1.*

(1)  $\implies$  (2): This follows from tightness of Borel probability measures, see [Kec95, 17.11].

(2)  $\implies$  (1): Let  $S \subseteq \Gamma$  be finite and let  $K \subseteq X$  be compact. Denote below by  $C(X, \text{Prob}(\Gamma))$  the set of continuous functions  $X \rightarrow \text{Prob}(\Gamma)$ , and define  $\Psi: C(X, \text{Prob}(\Gamma)) \rightarrow C(K)$  by

$$\Psi_p(x) = \sum_{\gamma \in S} \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1.$$

By measure-amenability and Lemma A.3.2, for every Borel probability measure  $\mu$  on  $K$ , we have

$$\inf_{f \in \text{im } \Psi} \int_K f \, d\mu = 0.$$

So by the Riesz representation theorem for  $C(K)$ , for every functional  $\varphi \in C(K)^*$ , we have

$$\inf_{f \in \text{im } \Psi} |\varphi(f)| = 0.$$

Thus by the Hahn-Banach separation theorem, we have

$$\inf_{f \in \text{Conv}(\text{im } \Psi)} \|f\|_\infty = 0,$$

where  $\text{Conv}(\text{im } \Psi)$  denotes the convex hull of  $\text{im } \Psi$ . Since

$$\Psi_{\sum_{i < k} \alpha_i p_i} \leq \sum_{i < k} \alpha_i \Psi_{p_i},$$

we have

$$\inf_{f \in \text{im } \Psi} \|f\|_\infty = 0,$$

so we are done.

Now suppose that  $X$  is  $\sigma$ -compact. It suffices to show (1)  $\implies$  (3). Write  $\Gamma = \bigcup_n S_n$  as an increasing union of finite subsets, and write  $X = \bigcup_n K_n$  as an increasing union of compact subsets. Then for each  $n$ , by topological amenability, there is some continuous  $p_n: X \rightarrow \text{Prob}(\Gamma)$  such that

$$\max_{\substack{\gamma \in S_n \\ x \in K_n}} \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1 < \frac{1}{n}.$$

Then  $(p_n)_n$  witnesses Borel amenability of  $\Gamma \curvearrowright X$ . □

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