

EQUIDECOMPOSITION IN CARDINAL ALGEBRAS

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ABSTRACT. Let Γ be a countable group. A classical theorem of Thorisson states that if X is a standard Borel Γ -space and μ and ν are Borel probability measures on X which agree on every Γ -invariant subset, then μ and ν are equidecomposable, i.e. there are Borel measures $(\mu_\gamma)_{\gamma \in \Gamma}$ on X such that $\mu = \sum_\gamma \mu_\gamma$ and $\nu = \sum_\gamma \gamma \mu_\gamma$. We establish a generalization of this result to cardinal algebras.

1. INTRODUCTION

In this paper, Γ will always denote a countable discrete group. Let X be a standard Borel Γ -space. A classical theorem of Thorisson [Tho96] in probability theory states that if x and x' are random variables on X , then the distributions of x and x' agree on the Γ -invariant subsets of X iff there is a shift-coupling of x and x' , i.e. a random variable γ on Γ such that γx and x' are equal in distribution. This characterization in terms of shift-coupling has been applied to various areas of probability theory including random rooted graphs [Khe18], Brownian motion [PT15], and point processes [HS13].

This theorem can be reformulated measure-theoretically as follows. Let μ and ν be Borel probability measures on X . Then μ and ν agree on every Γ -invariant set iff either of the following hold:

- (1) There is a Borel probability measure λ on $\Gamma \times X$ such that $s_*\lambda = \mu$ and $t_*\lambda = \nu$, where $s, t : \Gamma \times X \rightarrow X$ are the maps $s(\gamma, x) = x$ and $t(\gamma, x) = \gamma x$, and $s_*\lambda$ is the pushforward measure defined by $s_*\lambda(A) = \lambda(s^{-1}(A))$ (and similarly for $t_*\lambda$).
- (2) There is a Borel probability measure λ on the orbit equivalence relation $E_G^X := \{(x, y) \in X^2 : \exists \gamma[x = \gamma y]\}$ such that $s_*\lambda = \mu$ and $t_*\lambda = \nu$, where $s, t : E \rightarrow X$ are the maps $s(x, y) = x$ and $t(x, y) = y$ (see [Khe18, Theorem 1]).

By setting μ_γ to be the measure on X defined by $\mu_\gamma(A) := \mu(\{\gamma\} \times A)$, we see that μ and ν agree on every Γ -invariant set iff they are **equidecomposable**, i.e. there are Borel measures $(\mu_\gamma)_{\gamma \in \Gamma}$ on X such that $\mu = \sum_\gamma \mu_\gamma$ and $\nu = \sum_\gamma \gamma \mu_\gamma$, where $\gamma \mu_\gamma$ is the pushforward measure $\gamma \mu_\gamma(A) = \mu_\gamma(\gamma^{-1}A)$. In this paper, we show that this statement is an instance of a more general result about groups acting on generalized cardinal

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algebras, a concept introduced by Tarski in [Tar49], leading to a purely algebraic proof of the statement.

Remark 1.1. The original result in [Tho96] is stated for actions of locally compact groups, but it is not clear how to formulate an analogous theorem in the setting of cardinal algebras.

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2. PRELIMINARIES

A **generalized cardinal algebra (GCA)** is a set A equipped with a partial binary operation $+$, a constant 0 , and a partial ω -ary operation \sum subject to the following axioms, where we use the notation $\sum_n a_n = \sum(a_n)_n$:

- (1) If $\sum_n a_n$ is defined, then

$$\sum_n a_n = a_0 + \sum_{n \geq 1} a_n.$$

- (2) If $\sum_n (a_n + b_n)$ is defined, then

$$\sum_n (a_n + b_n) = \sum_n a_n + \sum_n b_n.$$

- (3) For any $a \in A$, we have $a + 0 = 0 + a = a$.

- (4) (Refinement axiom) If $a + b = \sum_n c_n$, then there are $(a_n)_n$ and $(b_n)_n$ such that

$$a = \sum_n a_n, \quad b = \sum_n b_n, \quad c_n = a_n + b_n.$$

- (5) (Remainder axiom) If $(a_n)_n$ and $(b_n)_n$ are such that $a_n = b_n + a_{n+1}$, then there is $c \in A$ such that for each n ,

$$a_n = c + \sum_{i \geq n} b_i.$$

These axioms imply in particular that \sum is commutative: if $\sum_n a_n$ is defined and π is a permutation of \mathbb{N} , then $\sum_n a_n = \sum_n a_{\pi(n)}$ (see [Tar49, 1.38]).

A **cardinal algebra (CA)** is a GCA whose operations $+$ and \sum are total. Cardinal algebras were introduced by Tarski in [Tar49] to axiomatize properties of ZF cardinal arithmetic, such as the cancellation law $n \cdot \kappa = n \cdot \lambda \implies \kappa = \lambda$. More recently, they have been used in [KM16] in the study of countable Borel equivalence relations.

Some examples of GCAs and CAs are as follows.

- [Tar49, 14.1] \mathbb{N} and \mathbb{R}^+ are GCAs under addition, where \mathbb{R}^+ is the set of non-negative real numbers.
- [Sho90, 2.1] If X is a measurable space, then the set of measures on X is a CA under addition.
- [Tar49, 15.10] Every σ -complete, σ -distributive lattice is a CA under join. In particular, for any set X , the power set $\mathcal{P}(X)$ is a CA under union.
- [Tar49, 17.2] The class of cardinals is a CA under addition (although strictly speaking, we require a CA to be a set).
- [Tar49, 15.24] Every σ -complete Boolean algebra is a GCA under join of mutually disjoint elements.
 - If X is a measurable space, then the collection $\mathcal{B}(X)$ of measurable sets is a GCA under disjoint union.
 - If (X, μ) is a measure space, then the measure algebra $\text{MALG}(X, \mu)$ is a GCA under disjoint union.

Every GCA is endowed with a relation \leq given by

$$a \leq b \iff \exists c[a + c = b]$$

This is a partial order with least element 0 (see the paragraph following [Tar49, 5.18]). Some examples of this partial order are as follows.

- In \mathbb{N} and \mathbb{R}^+ , \leq coincides with the usual order.
- In the CA of measures on X , $\mu \leq \nu$ iff $\mu(S) \leq \nu(S)$ for every measurable $S \subset X$.
- In the CA induced by a σ -complete, σ -distributive lattice, \leq is the partial order induced by the lattice, i.e. $a \leq b$ iff $a = a \wedge b$.
- For the class of cardinals, $\kappa \leq \lambda$ iff there is an injection $\kappa \hookrightarrow \lambda$, and the fact that this is a partial order is the Cantor-Schroeder-Bernstein theorem.

We say that $a \in A$ is the **meet** (resp. **join**) of a family $(a_i)_{i \in I}$, denoted $\bigwedge a_i$ (resp. $\bigvee a_i$), if it is the meet (resp. join) with respect to \leq . We write $a \perp b$ if $a \wedge b = 0$.

A **homomorphism** from a GCA A to a GCA B is a function $\phi : A \rightarrow B$ satisfying the following:

- (1) $\phi(a + b) = \phi(a) + \phi(b)$ whenever $a + b$ is defined.
- (2) $\phi(\sum_n a_n) = \sum_n \phi(a_n)$ whenever $\sum_n a_n$ is defined.
- (3) $\phi(0) = 0$.

An **action** of a countable group Γ on a GCA A is a group action $\Gamma \times A \rightarrow A$, denoted $(\gamma, a) \mapsto \gamma a$, such that for every $\gamma \in \Gamma$, the map $A \rightarrow A$ defined by $a \mapsto \gamma a$ is a homomorphism.

A Γ -**GCA** is a GCA A equipped with an action of a countable group Γ . An element a in A is Γ -**invariant** if $\gamma a = a$ for every $\gamma \in \Gamma$. We say that a and b in A are **equidecomposable** if there exist $(a_\gamma)_{\gamma \in \Gamma}$ in A such that $a = \sum_\gamma a_\gamma$ and $b = \sum_\gamma \gamma a_\gamma$.

The main theorem is as follows, where a GCA A is **cancellative** if for every $a, b \in A$, if $a + b = a$, then $b = 0$.

Theorem 2.1. *Let A be a cancellative Γ -GCA with binary meets, and let \sim be an equivalence relation on A such that the following hold:*

- (1) *Equidecomposable elements are \sim -related.*
- (2) *If $a \sim b$ and $a + c \sim b + d$, then $c \sim d$.*
- (3) *If $a \sim b$ and $a \perp \gamma b$ for every $\gamma \in \Gamma$, then $a = 0$ (this implies $b = 0$ by symmetry).*

Then $a \sim b$ iff a and b are equidecomposable.

We will show in [Section 4](#) that this implies Thorisson's theorem.

3. PROOF OF MAIN THEOREM

We turn to the proof of the main theorem.

Proof of Theorem 2.1. Fix an enumeration $(\gamma_n)_n$ of Γ . Suppose $a \sim b$. We define sequences (a_n) and (b_n) recursively as follows. Let $a_0 = a$ and $b_0 = b$. For the inductive step, choose a_{n+1} and b_{n+1} such that

$$\begin{aligned} a_n &= a_{n+1} + a_n \wedge (\gamma_n b_n) \\ b_n &= b_{n+1} + (\gamma_n^{-1} a_n) \wedge b_n \end{aligned}$$

By the Remainder axiom, there are some a_∞ and b_∞ such that for any n , we have

$$\begin{aligned} a_n &= a_\infty + \sum_{i \geq n} a_i \wedge (\gamma_i b_i) \\ b_n &= b_\infty + \sum_{i \geq n} (\gamma_i^{-1} a_i) \wedge b_i \end{aligned}$$

In particular, we have

$$\begin{aligned} a &= a_\infty + \sum_n a_n \wedge (\gamma_n b_n) \\ b &= b_\infty + \sum_n (\gamma_n^{-1} a_n) \wedge b_n \end{aligned}$$

Thus to show that a and b are equidecomposable, it suffices to show that $a_\infty = b_\infty = 0$.

Now $a_\infty \sim b_\infty$ by the second condition, since $a \sim b$ and $\sum_n a_n \wedge (\gamma_n b_n) \sim \sum_n (\gamma_n^{-1} a_n) \wedge b_n$ (by equidecomposability). Now for any n , we have $b_n \geq b_\infty + (\gamma_n^{-1} a_n) \wedge b_n$, and thus

$$\gamma_n b_n \geq \gamma_n b_\infty + a_n \wedge (\gamma_n b_n) \geq a_\infty \wedge (\gamma_n b_\infty) + a_n \wedge (\gamma_n b_n)$$

We also have

$$a_n \geq a_\infty + a_n \wedge (\gamma_n b_n) \geq a_\infty \wedge (\gamma_n b_\infty) + a_n \wedge (\gamma_n b_n)$$

Thus

$$a_n \wedge (\gamma_n b_n) \geq a_\infty \wedge (\gamma_n b_\infty) + a_n \wedge (\gamma_n b_n)$$

Since A is cancellative, we have $0 \geq a_\infty \wedge (\gamma_n b_\infty)$, i.e. $a_\infty \wedge (\gamma_n b_\infty) = 0$. Thus $a_\infty \perp \gamma b_\infty$ for every $\gamma \in \Gamma$. Thus by our hypothesis, we have $a_\infty = 0$ and $b_\infty = 0$. \square

4. APPLICATIONS

By a **finite measure** on a GCA A , we mean a homomorphism from A to \mathbb{R}^+ .

Corollary 4.1. *Let A be a Γ -GCA with countable joins and let μ and ν be finite measures on A . Then μ and ν agree on every Γ -invariant element of A iff they are equidecomposable.*

We recover Thorisson's theorem by setting $A = \mathcal{B}(X)$ (under disjoint union).

Corollary 4.2 (Thorisson, [Tho96, Theorem 1]). *Let X be a standard Borel Γ -space and let μ and ν be finite Borel measures on X . Then μ and ν agree on every Γ -invariant subset of X iff they are equidecomposable.*

To prove [Corollary 4.1](#), we need to define some more notions.

A **closure** of a GCA A is a CA \bar{A} containing A such that the following hold:

- (1) If a and $(a_n)_n$ are in A , then $a = \sum a_n$ in A iff $a = \sum_n a_n$ in \bar{A} .
- (2) A generates \bar{A} , i.e. for every $b \in \bar{A}$, there exist $(a_n)_n$ in A such that $b = \sum a_n$.

Proposition 4.3 ([Tar49, 7.8]). *Every GCA has a closure.*

Some examples of closures are as follows.

- $\bar{\mathbb{N}}$ is the set of extended natural numbers $\{0, 1, 2, \dots, \infty\}$.
- $\bar{\mathbb{R}}^+$ is the extended real line $[0, \infty]$.

The following is easy to verify.

Proposition 4.4. *If A is a GCA with closure \bar{A} and B is a CA, then every homomorphism $A \rightarrow B$ extends uniquely to a homomorphism $\bar{A} \rightarrow B$.*

Remark 4.5. This shows that the closure is left adjoint to the forgetful functor from the category of CAs to the category of GCAs, so in particular, the closure is unique up to isomorphism.

Let $\text{Hom}(A, B)$ denote the set of all homomorphisms from A to B .

Proposition 4.6. *Let A be a GCA. Then $\text{Hom}(A, \mathbb{R}^+)$ is a cancellative GCA with binary meets (under pointwise addition).*

Proof. By [Sho90, 2.1], $\text{Hom}(\overline{A}, \overline{\mathbb{R}^+})$ is a CA with binary meets, so $\text{Hom}(A, \overline{\mathbb{R}^+})$ is also CA with binary meets, since it is isomorphic to $\text{Hom}(\overline{A}, \overline{\mathbb{R}^+})$ by Proposition 4.4. Thus since $\text{Hom}(A, \mathbb{R}^+)$ is closed \leq -downwards in $\text{Hom}(A, \overline{\mathbb{R}^+})$, it is a GCA by [Tar49, 9.18(i)], and it has binary meets. The cancellativity of $\text{Hom}(A, \mathbb{R}^+)$ follows immediately from cancellativity of \mathbb{R}^+ . \square

We can now prove Corollary 4.1:

Proof of Corollary 4.1. $\text{Hom}(A, \mathbb{R}^+)$ is a cancellative GCA with binary meets, and it has a Γ -action given by $(\gamma\mu)(a) := \mu(\gamma^{-1}a)$. Define the equivalence relation \sim on $\text{Hom}(A, \mathbb{R}^+)$ by setting $\mu \sim \nu$ iff $\mu(a) = \nu(a)$ for every Γ -invariant $a \in A$. It suffices to check the conditions in Theorem 2.1. Conditions 1 and 2 are clear. For condition 3, suppose that $\mu \sim \nu$ and $\mu \perp \gamma\nu$ for every $\gamma \in \Gamma$, and fix $a \in A$. We must show that $\mu(a) = 0$. By [Tar49, 3.12], we have $\mu \perp \sum \gamma\nu$, and thus by [Sho90, 1.14] (which is stated for CAs, but whose proof works without modification for GCAs), we can write $a = b + c$ with $\mu(b) = 0$ and $(\sum \gamma\nu)(c) = 0$. Identifying ν with its extension $\overline{A} \rightarrow \overline{\mathbb{R}^+}$, we have $\nu(\sum \gamma c) = 0$. Thus $\nu(\bigvee \gamma c) = 0$, so since $\mu \sim \nu$, we have $\mu(\bigvee \gamma c) = 0$. Thus $\mu(c) = 0$, and thus $\mu(a) = \mu(b) + \mu(c) = 0$. \square

We also obtain a criterion for equidecomposability of subsets of a probability space. A **probability measure preserving (pmp)** Γ -action on a standard probability space (X, μ) is an action of Γ on (X, μ) by measure-preserving Borel automorphisms.

Corollary 4.7. *Let (X, μ) be a standard probability space with a pmp Γ -action and let $A, B \in \text{MALG}(X, \mu)$. Then A and B agree on every Γ -invariant measure $\ll \mu$ iff they are equidecomposable.¹*

This generalizes a well-known result (for instance, see [KM04, 7.10]) which says that if μ is ergodic, then A and B are equidecomposable iff $\mu(A) = \mu(B)$ (note that in this case, μ is the only Γ -invariant measure $\ll \mu$).

Corollary 4.7 will be obtained via a more general result about projections in von Neumann algebras; see Corollary 4.8 below.

We recall some notions from the theory of operator algebras; see [Bla06] for a standard reference. A **von Neumann algebra** is a weakly closed $*$ -subalgebra M of $B(H)$ containing the identity. An element $x \in M$ is **positive** if $x = yy^*$ for some $y \in M$, and the set of positive elements is denoted M_+ . There is a partial order on M defined by setting $x \leq y$ iff $y - x$ is positive. An element $p \in M$ is a **projection** if $p = p^* = p^2$, and the set of projections, denoted $P(M)$, is a complete lattice. Two projections p and q are **Murray-von Neumann equivalent**, written $p \sim_{\text{MvN}} q$, if there is some $u \in M$ such that $p = uu^*$ and $q = u^*u$. Then $P(M)/\sim_{\text{MvN}}$ is a complete

¹Ruiyuan (Ronnie) Chen has pointed out that this also follows from the Becker-Kechris comparability lemma [BK96, 4.5.1].

lattice. A projection p is **finite** if for any projection p' , if $p \sim_{\text{MvN}} p' \leq p$, then $p = p'$. A von Neumann algebra M is **finite** if 1_M is a finite projection. A **trace** on M is a map $\tau : M_+ \rightarrow \overline{\mathbb{R}^+}$ such that $\tau(mm^*) = \tau(m^*m)$, and a trace is **finite** if its image is contained in \mathbb{R}^+ . A trace is **faithful** if $\tau(m) = 0$ implies $m = 0$, and a trace is **normal** if it is weakly continuous.

If M is a von Neumann algebra, then $P(M)/\sim_{\text{MvN}}$ is a GCA under join of orthogonal projections [Fil65], and if M is finite, then this GCA is cancellative. A Γ -**action** on a von Neumann algebra M is an action of Γ on M by weakly continuous $(+, 0, \cdot, 1, *)$ -homomorphisms. Every von Neumann algebra M with a Γ -action gives rise to a Γ -GCA, and a trace τ on M is said to be Γ -**invariant** if $\tau(\gamma m) = \tau(m)$ for every $m \in M$ and $\gamma \in \Gamma$.

Corollary 4.8. *Let M be a finite von Neumann algebra with a Γ -action which admits a faithful normal finite Γ -invariant trace, and let $[p], [q] \in P(M)/\sim_{\text{MvN}}$. Then $[p]$ and $[q]$ agree on every finite Γ -invariant trace on M iff they are equidecomposable.*

Proof. Let $A = P(M)/\sim_{\text{MvN}}$, which is a cancellative Γ -GCA with binary meets. Now define the equivalence relation on A by setting $[p] \sim [q]$ if $[p]$ and $[q]$ agree on every Γ -invariant trace on M . It suffices to check the conditions in Theorem 2.1. Conditions 1 and 2 are clear. For condition 3, suppose that $[p] \sim [q]$ and $[p] \perp \gamma[q]$ for every $\gamma \in \Gamma$, and fix a faithful normal finite Γ -invariant trace τ on M . Then setting $\bar{p} = \bigvee \gamma p$, the map $m \mapsto \tau(\bar{p}m\bar{p})$ is a finite Γ -invariant trace on M . Since $\tau(\bar{p}q\bar{p}) = 0$ and $p \sim q$, we have $\tau(p) = \tau(\bar{p}p\bar{p}) = 0$. Thus $p = 0$. \square

Corollary 4.7 follows by applying this to $L^\infty(X, \mu)$.

Proof of Corollary 4.7. Let $M = L^\infty(X, \mu)$. This is a finite Γ -von Neumann algebra and μ induces a faithful normal finite Γ -invariant trace on M . Now $P(M)/\sim_{\text{MvN}}$ is isomorphic to $\text{MALG}(X, \mu)$ as a lattice (with Γ -action), so they give rise to isomorphic Γ -GCAs, and thus we are done by Corollary 4.8. \square

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