UC Berkeley Mock Putnam solutions

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1. Notice that $(p(x))^n$ has $3n+1$ coefficients, whose sum is $p(1)^n = 1.1^n$. So the average coefficient is $\frac{1.1^n}{3n+1}$. This goes to infinity, for example by l'Hospital's rule:

$$\begin{aligned}\lim_{n\to\infty} \frac{1.1^n}{3n+1} &= \lim_{x\to\infty} \frac{1.1^x}{3x+1} \\ &= \lim_{x\to\infty} \frac{1.1^x \log(1.1)}{3} = \infty.\end{aligned}$$

Since the largest coefficient is greater than or equal to the average, the largest coefficient goes to infinity as well.

Remark: Archit Kulkarni points out that a similar argument could be used even for $q(x) = 0.3x^3 - 0.4x^2 - 0.1x + 0.5$, for example by plugging in the norm-1 complex number $e^{2\pi i/3}$ instead of 1. This takes some extra care: at first, it only gives a lower bound for the largest coefficient in absolute value. However, a large negative coefficient would have to be canceled by some large positive coefficients, because the sum of the coefficients is $q(1)^n = 0.3^n > 0$.

2. **Solution 1.** No. Suppose the cube is originally located at coordinates $[0,1]^3$, and color the lattice points $\mathbb{Z}^3 \subset \mathbb{R}^3$ with a checkerboard; i.e. color (x,y,z) white if $x+y+z$ is even and black if it is odd. Then with each move, each corner of the cube moves from one lattice point to another lattice point of the same color. The desired transformation would move every corner to a lattice point of the opposite color, so it is impossible.

Solution 2. Suppose for contradiction that it is possible. We use two different parity arguments, first to show that it must take an even number of moves, and then to show it must take an odd number of moves. For the former, paint the plane with a checkerboard, with the square initially resting on a light square. Every move goes from a light square to a dark one or vice versa, so we need an even number of moves to return to the initial square. (Or alternatively: the number of forward rolls equals the number of backward rolls, and similarly for right and left, so the sum of all four numbers is even.)

To show the number of moves must be odd, look at the cube in a reference frame that moves, but does not rotate, with the cube. In this reference frame, each move is a four-cycle of faces, which is an odd permutation. We want to do a different four-cycle of faces, which is also an odd permutation. But an odd permutation cannot be written as a product of an even number of odd permutations, so we need an odd number of moves.

This is a contradiction.

Solution 3. Instead of the second parity argument of solution 2, we color the vertices of the cube white and black in such a way that adjacent vertices have opposite colors. Then every move sends every white corner to a black corner and vice versa, in the reference frame of the cube. The desired operation switches the colors, so it must be done in an odd number of moves.

Solution 4. Instead of the second parity argument of solution 2, we say that the faces on the top, front, and right are *visible*, and we label each face with an integer, so that the originally visible faces are odd numbers and the rest are even. Consider the sum of the numbers on the visible faces as the cube is rotated. Each move keeps two of those three faces visible and replaces one with its opposite. Since opposite faces have numbers of opposite parities, the sum of the visible numbers changes parity with every move. It started odd and must finish even, so we need an odd number of moves.

Solutions 5-7. Several other permutation parity arguments can replace the second half of solution 2. For example, the rotations of a cube can act as permutations on the three pairs of opposite faces, the four pairs of opposite corners, or the 12 edges; each 90-degree rotation is an odd permutation of any of these things. In particular, the possible moves and the target permutations are all odd permutations, so an odd number of moves are needed.

Remark: The group of rotations of a cube is isomorphic to the symmetric group S_4, and one isomorphism is given by its action on the four pairs of opposite corners. The other permutation parity arguments we used exhibit homomorphisms $S_4 \to S_n$ for $n = 6, 3, 12$, which send even permutations to even permutations and odd to odd. The homomorphism $S_4 \to S_3$ is surjective, and the others are injective.

3. There are $n! - 1$ impossible labelings. First suppose that all n of the labels are distinct. Then any move will leave two balls with different labels in the same box, and they will always remain together. So in this case, your goal is possible if and only if it has already been achieved; the $n! - 1$ non-identity permutations are impossible.

   On the other hand, we claim that your goal can always be achieved if at least two balls have the same label. To achieve it, begin by consolidating balls with matching labels: for each label that appears on more than one ball, pour all balls with that label into one of their initial boxes. Note that once we have done this (or even made one move), there will always be at least one empty box available to us, namely the one we most recently poured from.

   After this, use the following procedure until all balls are in the correct place. Choose a box, A, that contains balls that belong in another box, B. If B is empty, pour A into B. Otherwise, find an empty box C, pour B into C, and then pour A into B. Now all balls labeled B, and only them, will be in B. This means we can ignore box B and repeat inductively with the remaining $n-1$ boxes until we accomplish our goal.

4. **Solution 1.** The answer is $2n - 1$. Clearly it is not less than that: G must have a set of

n vertices all connected to each other, and a set of n vertices all disconnected from each other, and at most one vertex can lie in both sets. To construct such a G with $2n-1$ vertices, we induct on n. For $n=1$, a single vertex with no edges works. Then suppose we are given a graph G on vertex set $V = \{v_1, \ldots, v_{2n-1}\}$ that works for n. Build a new graph G' on vertex set $V' = V \cup \{w_0, w_1\}$, where w_0 is connected to nothing and w_1 is connected to all v_i. Then for any $0 \le k \le \binom{n}{2}$, we can find an induced subgraph with $n+1$ vertices and k edges, consisting of n vertices from V along with w_0. For any $n \le k \le \binom{n+1}{2}$, we can choose n vertices from V with $k-n$ edges among them, and add w_1 to them. Since $\binom{n}{2} \ge n-1$ for all n, this exhausts all choices of k. So $2n-1$ vertices suffices for all n by induction.

Solution 2. As proved in solution 1, no such G exists on fewer than $2n-1$ vertices. We give an explicit description of a graph on $2n-1$ vertices that works. Let G have vertices $a_0, \ldots, a_{n-1}$ and $b_1, \ldots, b_{n-1}$, with all a_i disconnected, all b_j connected, and a_i connected to b_j if and only if $j > i$.

Given $0 \le k \le \binom{n}{2}$, we choose $m \ge 1$ such that $\binom{m}{2} \le k < \binom{m+1}{2} = \binom{m}{2} + m$, and we set $\ell = k - \binom{m}{2}$. If $k = \binom{n}{2}$, then we choose the n vertices $a_0, b_1, \ldots, b_{n-1}$; otherwise, we have $m \le n-1$. Then the $n-1$ vertices $b_1, \ldots, b_m, a_{m+1}, \ldots, a_{n-1}$ have exactly $\binom{m}{2}$ edges among them, namely the ones among the b_j. Adding $a_{m-\ell}$ (which is valid because $0 \le \ell < m$) gives us an additional ℓ edges, from this vertex to $b_{m-\ell+1}, \ldots, b_m$, so we have found n vertices joined by exactly $\binom{m}{2} + \ell = k$ edges, as desired.

Solution 3. As before, no such G exists on fewer than $2n-1$ vertices. To construct one with $2n-1$ vertices, let the vertex set consist of the integers $1-n, \ldots, n-1$, and draw an edge between two numbers if their sum is positive. Note that this graph contains all edges among $\{0, \ldots, n-1\}$ and none among $\{1-n, \ldots, 0\}$.

To get an induced subgraph with intermediate numbers of edges, we use the following lemma: if there are k vertices among $\{a_1, \ldots, a_i, ..., a_n\}$, then there must be either k or $k+1$ vertices among $\{a_1, \ldots, a_i + 1, \ldots, a_n\}$. (Here we are assuming that all vertices in both sets are distinct.) Proof: the only possible change is that $a_i + 1$ is connected to $-a_i$, and a_i isn't. So the number of edges increases by 1 if some a_j equals $-a_i$, and stays constant otherwise.

To finish the proof, begin with the set $\{1-n, \ldots, 0\}$, and slide the vertices over to $\{0, \ldots, n-1\}$, moving one vertex at a time one step at a time to the right. This will produce a sequence of induced subgraphs, starting with 0 edges and ending with $\binom{n}{2}$ edges, and attaining every intermediate value at least once.

Remark: The graphs constructed in the second and third solutions are isomorphic, by identifying a_i with $-i$ and b_j with j. They are also isomorphic to the graph constructed recursively in the first solution, where w_0 corresponds to a_{n-1} and w_1 corresponds to b_{n-1}. But there are other graphs that work: for example, in the first solution, we are free to choose at each stage of the recursion whether there is an edge between w_0 and w_1.

5. **Solution 1:** Bob can win. We claim that Bob can play in such a way that there is equally

much blue and red after each of his moves. This is clearly true for the first move: Bob avoids overlapping Alice's circle.

Suppose by induction that Bob has played his first $n > 0$ moves and maintained equality of red and blue so far. If Alice plays anywhere and then Bob plays on top of her move, then there will be at least as much blue as red, since nothing has been painted red that wasn't red before, and everything that was blue before remains blue. If instead Bob repeats his own nth move, the exact opposite will be true, so there will be at least as much red as blue. It follows by the intermediate value theorem that Bob can equalize the colors, since (blue area − red area) is a continuous function of the point in the plane at which Bob places his circle. So by induction, Bob can keep the areas equal after any number of moves, and thereby win the game.

Remark: This strategy allows Bob to win if the game is played on any convex region in the plane, provided that Bob has enough room to avoid overlapping Alice's first move. For example, a circle of radius 3 or a square of side length $2\sqrt{2}+2$ is enough. On a smaller board, Alice wins by playing all of her moves in the center.

Remark: Some people noticed that every move increases the player's "margin" (i.e. player's area minus opponent's area) by some number in $[0, 2\pi]$, and claimed that Bob can always simply match the amount Alice increases her margin by. This is true by the argument above, but not obvious: Bob can always increase his margin by any number in $[\pi, 2\pi]$, but only sometimes by numbers in $[0, \pi)$. For example, he can only keep his margin constant if he has a solid blue circle of radius 1 somewhere on the board after Alice's move, and Alice can prevent this if she wants.

Solution 2: Bob can win. We claim by induction that Bob can force the following situation to hold for each n: After Bob's nth move, the circles played so far can be arranged into some number of clusters such that

(i) the clusters come in pairs, where a pair of clusters differ by some translation in the plane and otherwise look identical with colors switched; and

(ii) no two circles in different clusters have their centers within 2^{2018-n} units of each other.

In particular, if Bob can achieve this, then no two circles in different clusters will overlap after 2017 moves, by condition (ii), so Bob will win by condition (i). To achieve it on move 1, Bob plays at least 2^{2017} units away from Alice. Afterwards, suppose the conditions hold after Bob's nth move, and define two circles to be *close* on move $n+1$ if their centers are within $2^{2018-(n+1)}$ units apart. Then if Alice's $(n+1)$st move is not close to any existing circle, Bob plays a circle that is also not close to any existing circle. This clearly satisfies conditions (i) and (ii), where the clusters are as before plus a new pair of singleton clusters.

On the other hand, if Alice's $(n+1)$st move is close to some existing circle, it can only be close to one existing cluster C, by the inductive hypothesis and the triangle inequality. Bob plays in the corresponding location relative to that cluster's pair D (which is different from C). Bob's new circle is not close to Alice's, because they can be translated to a pair of existing circles in C and D, which are at least 2^{2018-n} units apart by the inductive hypothesis. Nor is Bob's new circle close to any other existing circle outside D, again by the inductive hypothesis and the triangle inequality. So if we declare Alice's and Bob's new moves to be part of clusters C and D respectively, then condition (ii) is satisfied. Since Bob matched the location of Alice's new move, and no clusters other than C and D were overlapped, condition (i) is also satisfied. This completes the induction, so Bob can win.

Remark: Bob can also replace "translation" in condition (i) by "isometry" (i.e. allowing rotations and reflections), at the cost of replacing powers of 2 by powers of 3. Otherwise, Bob's new move may be close to Alice's new move.

Remark: This solution is trickier than one might expect. If the game were played on the disjoint union of two planes, Bob would easily win by matching all of Alice's moves on the other plane. As it is, Bob might reasonably try playing (e.g.) $4 \cdot 2017$ units away from Alice's first move, thinking that this is far enough that the two moves "can't affect each other", and that he can simply match whatever Alice plays thereafter. However, Alice can complicate matters by always playing halfway between two existing moves. Within approximately $\log_2(4 \cdot 2017) \approx 13$ moves, Alice will be able to create an overlap that Bob won't be able to mirror. In order to maintain symmetry, Bob needs to play exponentially far away.

6. **Solution 1:** (Essentially the one found by four competitors.) Note that a_n is odd for $n > 1$, since $2^{a_{n-1}}$ is even and $3^{a_{n-1}}$ is odd. We claim that if N satisfies the given condition, then $2^N + 3^N$ does too. To prove this, assume that $N \mid a_n$ for all sufficiently large n. Then for sufficiently large n, we have $a_{n+1} = 2^{a_n} + 3^{a_n} = 2^{kN} + 3^{kN}$ for $k = a_n/N$. Since k is odd (it divides the odd number a_n, $n > 1$), this is divisible by $2^N + 3^N$.

Since $N = 1$ works and $2^N + 3^N > N$ for all N, this gives an infinite sequence of distinct integers $1, 2^1 + 3^1 = 5, 2^5 + 3^5 = 275, \ldots$ that work.

Solution 2: We claim that $N = 5^k$ works for all k. First observe that all terms beyond a_0 satisfy $a_n \geq 2$, so the following congruences hold for all $n > 0$:

$$\begin{aligned} a_{n+1} &= 2^{a_n} + 3^{a_n} \equiv 0 + 1 \equiv 1 \pmod{2}; \\ a_{n+2} &= 2^{a_{n+1}} + 3^{a_{n+1}} \equiv 0 + 3 \equiv 3 \pmod{4}; \\ a_{n+3} &= 2^{a_{n+2}} + 3^{a_{n+2}} \equiv 3 + 2 \equiv 0 \pmod{5}, \end{aligned}$$

where each line follows from the preceding one. We use this as a base case for induction on k. Suppose that $k \geq 1$ and all but finitely many a_n are divisible by 5^k. We claim that if n is large enough so that $5^k \mid a_n$, then $5^{k+1} \mid a_{n+1}$; this implies that $N = 5^{k+1}$ works.

To prove our claim, first observe that since $\varphi(5^{k+1}) = 4 \cdot 5^k$, Euler's theorem implies that $2^{4 \cdot 5^k} \equiv 3^{4 \cdot 5^k} \equiv 1 \pmod{5^{k+1}}$. Given that $5^k \mid a_n$, we get that $4 \cdot 5^k \mid 4a_n$ and therefore $2^{4a_n} \equiv 3^{4a_n} \equiv 1 \pmod{5^{k+1}}$. So 2^{a_n} and 3^{a_n} are both fourth roots of unity modulo 5^{k+1}. But one can show using Hensel's lemma that there are exactly four fourth roots of unity modulo any 5^{k+1}: ± 1, and a pair of integers mod 5^{k+1} that we call $\pm\zeta$, where $\zeta \equiv 2$

(mod 5). But since $a_n \equiv 3 \pmod 4$ for n large enough, we have

$$\begin{aligned} 2^{a_n} &\equiv 2^3 \equiv 3 \pmod 5 \text{ and} \\ 3^{a_n} &\equiv 3^3 \equiv 2 \pmod 5, \end{aligned}$$

since $2^4 \equiv 1 \pmod 5$ by Fermat's little theorem. So in fact 2^{a_n} and 3^{a_n} are respectively congruent to $-\zeta$ and $\zeta \pmod{5^{k+1}}$. So $2^{a_n} + 3^{a_n} \equiv 0 \pmod{5^{k+1}}$, and we are done by induction.

Remark: One can also prove the inductive step above by showing that all but the first and last terms in the binomial expansion of $(2+3)^{a_n}$ are divisible by 5^{k+1}, as is 5^{a_n} itself.

Remark: Ben Castle, who miscalculated mod 25 while test-solving this problem, discovered that 11^n also works for all n. In fact, a version of solution 2 can be used to show that for any (necessarily odd) prime $N = p$ that works, all p^k must also work. This comes from the fact that the ring $\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z}$ of p-adic integers contains a full set of $(p-1)$st roots of unity, called Teichmüller representatives, with exactly one in each nonzero residue class modulo p; 2^{a_n} and 3^{a_n} will converge on some pair of additive inverses among them.