UC Berkeley Mock Putnam exam

September 30, 2017

Instructions: Write your solutions on the answer sheets provided, labeling each page with your exam number. If you need extra answer sheets, use separate sheets for each problem. *For full credit, you must give carefully argued proofs, written legibly and in the correct logical order.*

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1. Let $p(x) = 0.3x^3 + 0.4x^2 - 0.1x + 0.5$, and let a_n denote the largest coefficient of $(p(x))^n$. (For example, $a_1 = 0.5$, $a_2 = 0.41$, and $a_3 = 0.321$.) Show that $\lim_{n\to\infty} a_n = \infty$.

2. A unit cube sits on a plane. You can roll the cube without slipping in any of the four directions parallel to its sides; for example, if it rolls to the right, its center will travel one unit to the right and its top face will become its new right face. After a finite number of these moves, is it possible to return the cube to its original location, but rotated 90 degrees around its vertical axis?

3. You are given n boxes labeled 1 to n in order, each containing exactly one ball. The balls are labeled independently with integers in $\{1, \dots, n\}$; different balls may have the same label. You are allowed to perform a sequence of moves, each consisting of pouring the contents of one box completely into another box. Your goal is to finish with each ball sitting in the box that matches the ball's label. How many initial labelings are there from which your goal cannot be achieved?

4. For each positive integer n, find the smallest possible number of vertices of a graph G with the following property: for each $0 \leq k \leq \binom{n}{2}$, there exists a choice of n vertices of G such that G contains exactly k edges among the chosen vertices.

5. Alice and Bob play the following game, taking turns with Alice moving first. Each turn consists of painting a solid disk of radius 1 somewhere on the plane; Alice uses red paint and Bob blue. If a new circle overlaps an old circle, the old paint gets covered and is no longer visible. The game ends when they have played 2017 moves each. Bob wins if the total area painted blue at the end equals the total area painted red, and Alice wins otherwise. Decide who wins with correct play, and find a winning strategy for them.

6. Define a sequence recursively by $a_0 = 0$ and $a_{n+1} = 2^{a_n} + 3^{a_n}$ for $n \geq 0$. Prove that there exist infinitely many integers N with the property that all but finitely many a_n are divisible by N.