Math 191 Homework 8: Number theory

Due: Monday, October 30, 2017

The problems are weighted by (approximate) difficulty. Solve at least 13 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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1 point

1. (1988 B1) A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

2. Find all integers n such that $n \equiv 1 \pmod 3$, $n \equiv 2 \pmod 5$, and $n \equiv 3 \pmod 7$. (Try to do this as efficiently as possible.)

3. An integer is called *squarefree* if it is not divisible by any squares other than 1. Prove that there exist 100 consecutive integers none of which are squarefree.

2 points

4. (2005 A1) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

5. (1989 A1) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

6. Prove that the equation $a^2 + b^2 = 3c^2$ has no integer solutions other than $a = b = c = 0$.

7. (2013 A2) Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \dots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4, 2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

8. Show that the product of four consecutive positive integers cannot be a square.

3 points

9. What is the greatest common divisor of the numbers $n^{2017} - n$, as n runs over all integers?

10. Prove that for all $n > 0$, there are exactly two solutions to $x^2 \equiv -1 \pmod{5^n}$. (This is a special case of what's called Hensel's lemma, but don't just quote Hensel's lemma—prove it yourself.)

11. Prove that there are infinitely many integer solutions to $a^2 - 2b^2 = 1$. (This is a special case of Pell's equation, but prove it yourself. Possible hint: look at $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$.)

12. Suppose that n is a positive integer such that one of n and $n+1$ is a power of 2, and the other is a power of 3. Prove that $n = 1, 2, 3,$ or 8. (Again, do this by yourself, without quoting any big theorems like Mihăilescu's proof of Catalan's conjecture.)

13. (Related to 1985 A4 and 1997 B5.) Let m and n be any positive integers. Prove that the sequence $(n, n^n, n^{n^n}, \dots)$ is eventually constant modulo m.

14. (2014 B3) Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m + n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

4 points

15. (2008 B4) Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \dots, h(p^2 - 1)$ are distinct modulo p^2. Show that $h(0), h(1), \dots, h(p^3 - 1)$ are distinct modulo p^3.

16. (2010 A4) Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

5 points

17. (1991 B5) Let p be an odd prime and let $\mathbb{Z}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : y \in \mathbb{Z}_p\}?$$