Math 191 Homework 7: Polynomials

Due: Monday, October 23, 2017

The problems are weighted by (approximate) difficulty. Solve at least 14 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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1 point

1. (1986 A1) Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

2. (1985 B1) Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

3. (2007 B1) Let f be a nonconstant[1] polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

2 points

4. (Lagrange interpolation) Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ with the a_i distinct. Suppose $p(x)$ is a polynomial of degree less than n with $p(a_i) = b_i$ for all i. Prove that

$$\begin{aligned} p(x) = {} & b_1 \frac{(x - a_2)(x - a_3) \cdots (x - a_n)}{(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n)} + b_2 \frac{(x - a_1)(x - a_3) \cdots (x - a_n)}{(a_2 - a_1)(a_2 - a_3) \cdots (a_2 - a_n)} \\ & + \cdots + b_n \frac{(x - a_1)(x - a_2) \cdots (x - a_{n-1})}{(a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1})}. \end{aligned}$$

5. (1985 B2) Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{\mathrm{d}}{\mathrm{d}x} f_{n+1}(x) = (n + 1) f_n(x + 1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

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[1] This hypothesis was omitted in the original problem.

6. Let $f : \mathbb{F}_p \to \mathbb{F}_p$ be any function. Prove that f can be expressed as a polynomial function

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

with coefficients $a_i \in \mathbb{F}_p$. (Here $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field of p elements.)

7. (1986 A2) What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

3 points

8. Let $f(x)$ be a real-valued polynomial with $f(x) \geq 0$ for all x. Must $f(x)$ be a sum of (finitely many) squares of polynomials?

9. (Evan O'Dorney) Find two real numbers such that their sum is 2 and the sum of their fifth powers is 22. Give an *exact* answer.

10. (a) Let $f(x)$ be a real-valued polynomial that is bounded below. Must f attain a global minimum?

    (b) Same question for $f(x, y)$ a polynomial function $\mathbb{R}^2 \to \mathbb{R}$.

11. (From Kannan Soundararajan) Two players, A and B, play the following game. A thinks of a polynomial with non-negative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has another such try. Can B *guarantee* a win?

5 points

12. (2010 B4) Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

13. (2008 A5) Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

14. (Alexander Bertoloni-Meli) Let $f(x, y)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$. Suppose that for every constant y_0, $f(x, y_0)$ is a polynomial in x, and for every constant x_0, $f(x_0, y)$ is a polynomial in y. Must $f(x, y)$ be a polynomial?