Math 191 Homework 6: Linear algebra

Due: Monday, October 16, 2017

The problems are weighted by (approximate) difficulty. Solve at least 13 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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2 points

1. (2008 A2) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

2. (2014 A2) Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is $1/\min(i,j)$ for $1 \le i,j \le n$. Compute $\det(A)$.

3. (2003 B1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

   holds identically?

4. (For people who haven't worked with field extensions before.) If F is a field and $E \subset F$ is a subfield (i.e. a subset that is a field with the same $+$ and $\cdot$), then show that F is a vector space over E. Use this to show that the field of 4 elements cannot be a subfield of the field of 8 elements.[1]

5. (Fall 2014 Berkeley prelim exam 6B) Show that there is a sequence of 4×4 matrices A_n with real entries, which converges to

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

   and such that each A_n has 4 distinct real eigenvalues, two of which are positive and two negative.

[1] Up to isomorphism, there is exactly one field with p^n elements for each prime p and integer $n \ge 1$, and no other finite fields. The ones of order p are $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$; in general, $\mathbb{F}_{p^n}$ is more complicated but contains $\mathbb{F}_p$ as a subfield. You don't need any of this to solve the problem, but you may end up proving some of it anyway.

6. (1995 B3) To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

7. (1994 A4) Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

3 points

8. (Fall 2015 Berkeley prelim exam 7A) It is a corollary to the Jordan canonical form theorem that $n \times n$ matrices in Jordan canonical form, all of whose eigenvalues are zeroes, are similar if and only if the sizes of their Jordan blocks coincide (up to permutations). Prove this directly, without using the Jordan canonical form theorem.

9. S is a finite collection of real numbers, not necessarily distinct. If any element of S is removed, then the remaining numbers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal. (You may assume the corresponding fact for integers, which was on HW 2.)

10. (2011 A4) For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

11. A k-dimensional parallelepiped in n-dimensional space (with $k \leq n$) is generated by the vectors $v_1, \ldots, v_k$. Show that its volume is $\sqrt{\det A}$, where A is the $k \times k$ matrix with entries $A_{ij} = v_i \cdot v_j$.

4 points

12. (2006 B4) Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum of $Z(V)$ over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k.

13. (2011 B4) In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1. Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.