Math 191 Homework 5: Geometry and complex numbers

Due: Monday, October 9, 2017

The problems are weighted by (approximate) difficulty. Solve at least 14 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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1 point

1. (1998 A1) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

2. (Alexander Givental, 2016 mock Putnam #1) Three pedestrians are traveling with constant speeds along three straight roads. Show that if at the starting moment they were not on the same line, then there are at most two moments of time when they can turn out on the same line.

2 points

3. Use complex numbers (e.g. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$) to prove the formulas $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, $\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$, and $\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$.

4. (1996 A2) Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

5. (2015 A1) Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

6. (1998 B2) Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

7. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

8. (Olga Holtz, 2004 mock Putnam #2) Show that a necessary and sufficient condition for three points a, b, and c in the complex plane to form an equilateral triangle is that

$$a^2 + b^2 + c^2 = bc + ca + ab.$$

3 points

9. (2016 B3) Suppose that S is a finite set of points in the plane such that the area of triangle ΔABC is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

10. (Olga Holtz, 2004 mock Putnam #3) Find all nonconstant polynomials p with real coefficients such that $p(x^2) = p(x) \cdot p(x-1)$ for all $x \in \mathbb{R}$.

11. (2000 A3) The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

12. (Kousuke Shimoda) The convex quadrilateral $ABCD$ has angles $CAB = 30°$, $CBD = 96°$, and $BCA = ACD = 24°$. Find the angle CAD.
(Picture: `http://www.thepicta.com/media/1463332393595118026_2284283487`.)

13. (2012 B2) Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

14. You are given n pipes (segments in $\mathbb{R}^3$ of length 1) that can be connected with each other by meeting at their endpoints in a right angle. For example, 4 pipes could form a square, or 6 pipes could form an L-shaped closed loop—but there is no requirement that the pipes be parallel to the coordinate axes. Is there some odd number n such that n pipes can form a closed loop in $\mathbb{R}^3$?

4 points

15. (2013 A5) For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \leq i < j < k \leq m)$ is said to be *area definite* for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\Delta A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1, a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.