Math 191 Homework 4: Probability and combinatorics

Due: Monday, September 25, 2017

The problems are weighted by (approximate) difficulty. Solve at least 14 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

---

1 point

1. (2003 A1) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

2. (2004 A1) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

3. If π is a permutation of $S = \{1, \ldots, n\}$, chosen randomly with uniform probability, what is the expected number of fixed points of π? (A *fixed point* is an element $s \in S$ with $\pi(s) = s$.)

4. (1985 A1) Determine, with proof, the number of ordered triples $\{A_1, A_2, A_3\}$ of sets which have the property that

   (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

   (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$,

   where $\emptyset$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, and d are nonnegative integers.

2 points

5. (2004 B2) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!n!}{m^m n^n}.$$

6. (Putnam and Beyond #912) What is the probability that a randomly chosen permutation of $\{1, 2, ..., n\}$ has 1 and 2 in the same cycle?

3 points

7. Let $G = (V, E)$ be a graph with n vertices and e edges. Show that G contains a bipartite subgraph with at least $e/2$ edges.

8. (2006 A4) Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

   (i) $\pi(k) > \pi(k + 1)$ for $k = 1$;

   (ii) $\pi(k - 1) < \pi(k)$ and $\pi(k) > \pi(k + 1)$ for $1 < k < n$;

   (iii) $\pi(k - 1) < \pi(k)$ for $k = n$.

   (For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

9. (2007 A3) Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k + 1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

10. (2001 A2) You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

11. (Spring 1958 A3) A sequence of real numbers $a_1, a_2, \ldots$ is chosen uniformly and independently at random from the interval $[0, 1]$. If n is the smallest number such that $\sum_{i=1}^{n} a_i > 1$, show that the expected value of n is e.

4 points

12. Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$. Show that there exist $\epsilon_1, \ldots, \epsilon_n, \epsilon'_1, \ldots, \epsilon'_n = \pm 1$ such that

$$\left| \sum_{j=1}^{n} \epsilon_j v_j \right| \leq \sqrt{n} \qquad \text{and} \qquad \left| \sum_{j=1}^{n} \epsilon'_j v_j \right| \geq \sqrt{n}.$$

13. (2003 A5) A *Dyck n-path* is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin O and never dips below the x-axis. A *return* is a maximal sequence of contiguous downsteps that terminates on the x-axis. Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n - 1)$-paths. (See illustration: `http://kskedlaya.org/putnam-archive/2003.pdf`.)

14. (2004 A5) An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.