Math 191 Homework 3: Invariants and monovariants

Due: Monday, September 18, 2017

The problems are weighted by (approximate) difficulty. Solve at least 15 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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1 point

1. 2017 people enter a rock-paper-scissors tournament. They are paired randomly, with one player getting a bye in any round that starts with an odd number of competitors, and they keep playing single-elimination rounds until only the winner remains. How many games will be played?

2. (Brilliant.org, adapted from NIMO) Alice and Bob have a large chocolate bar, in the shape of a 10×10 grid. Each turn, a player may either eat an entire bar of chocolate, or break any chocolate bar into two smaller rectangular chocolate bars along a grid line. The player who moves last loses. Who wins this game?

3. Two people play a variant of chess with one knight each and no other pieces. The knights move according to usual chess rules, and the goal is to capture your opponent's knight. Prove that given any starting position, one of the players can't lose no matter how hard they try to. Which player is it?

4. Find a sequence of positive integers $a_1, \ldots, a_n$ with $\sum_i a_i = 100$ and the largest possible value of $\prod_i a_i$.

2 points

5. The numbers 1, 2, ..., 100 are written on a chalkboard. You are allowed to erase two numbers, a and b, at a time, and replace them with $ab + a + b$. You repeat this process until only one number remains. Prove that this number does not depend on your choices, and find its value.

6. Some people are in a building with several rooms. Each minute a person leaves a room and moves to another room that has at least as many people (counted immediately before they move). Show that eventually all of the people are in a single room.

7. You are given an $m \times n$ array of real numbers. You are allowed to flip all of the signs on any row or any column. Prove that after finitely many such moves, you can force all row sums and all column sums to be nonnegative.

8. Start with the set $\{3, 4, 12\}$. You are allowed to replace any two numbers, a and b, with the new numbers $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you transform the set into $\{4, 6, 12\}$ in a finite number of moves?

9. Consider an infinite first-quadrant checkerboard that is initially empty except for three checkers placed in an L shape in the lower left corner. You are allowed to perform the following move: remove any one checker from the board and replace it with two checkers, one on the square immediately above it and one on the square immediately to the right of it. You can never put more than one checker on the same square. Is it possible to remove all the checkers from the three originally filled squares?

10. You are given a chain consisting of 1001 links. You have a bolt cutter that will cut any one link of the chain, thereby destroying it and separating the rest into two smaller chains. However, you don't want to cut a link right at the end of a length of chain, since that would be wasteful. How many wasteful cuts must you make in order to cut the chain into pieces with exactly three links each?

11. Three bugs start out at points $(0, 0), (3, 0)$, and $(0, 2)$. They move one at a time, in any order. Each bug can only move in a direction parallel to the line containing the other two bugs. Can two of the bugs switch places while the third ends up where it started? Can they end up at $(1, 1), (6, 2), (3, 4)$?

3 points

12. Given n red points and n blue points in the plane, no three collinear, show that we can draw n non-intersecting line segments, each having one red endpoint and one blue endpoint. (For this problem, sharing an endpoint counts as intersecting.)

13. Twelve people sit around a table, one of them holding 12 cards and the rest holding nothing. The cards are passed around according to the following rule: at any time, a person holding two or more cards is allowed to simultaneously pass one card to the left and one to the right. Will the game end, or will there will always be someone holding at least two cards?

14. You have a shuffled deck of cards, labeled 1 through n. Whenever the card labeled k is on top, you pick up the top k cards and reverse their order. Prove that eventually the 1 card will be on top.

15. (2008 A3) Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k , and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

16. On an $n \times n$ board there are n^2 squares, $n - 1$ of which are infected. Each second, any square that is adjacent to at least two infected squares becomes infected. Show that at least one square always remains uninfected.

6 points

17. There are n books on a bookshelf, numbered from 1 to n but placed in a random order. Hoping to put the books in order, you pull out a random book that is not in the correct position and move it to its correct position, sliding all of the in-between books one spot back towards the gap. Prove that this process will always arrange the books correctly in a finite number of steps.