Math 191 Homework 2: Pigeonhole and parity

Due: Monday, September 11, 2017

The problems are weighted by (approximate) difficulty. Solve at least 12 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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1 point

1. Prove that if five points are chosen in the interior of a unit square, some two of them must be at most $\frac{\sqrt{2}}{2}$ units apart.

2. (2010 A1) Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

3. (2012 A1) Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

4. Let $P(x)$ be a polynomial with integer coefficients and degree at most n, where $n > 1$. Suppose that $|P(x)| < n$ for all $|x| < n^2$. Show that P is constant.

5. Suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients, such that a_0 and $\sum_{i=0}^{n} a_i$ are both odd. Prove that $f(x) = 0$ has no integer solutions.

2 points

6. (2013 A1) Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

7. Find a simple criterion to determine whether a given integer can be written in the form $m^2 - n^2$, with $m, n \in \mathbb{Z}$, and use it to tell how many of the integers $10^9 + 1, 10^9 + 2, \ldots, 10^9 + 100$ can be written in this form.

8. (Paul Erdős) Prove that among any $n + 1$ numbers in $[1, 2n]$, one must divide another.

9. One hundred prisoners, each wearing either a white or a black hat, stand in a line. They are all facing forward, so that they can see the colors of the hats in front of them, but not their own. The prison warden goes from the back of the line to the front, asking each prisoner the color of their own hat. Everyone can hear everyone else's answers, but they

are not told whether those answers were correct or not. The prisoners will be released if and only if all 100 of them guess correctly. Find a strategy that gives them a 50% chance of success.

10. (2006 B2) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

11. Let α be an irrational real number. Prove that as n ranges over $\mathbb{Z}$, the set $\{n\alpha\}$ is dense mod 1; i.e. for any $0 \leq a < b \leq 1$, there exists n such that the fractional part of $n\alpha$ is between a and b.

3 points

12. (1973 B1) S is a finite collection of integers, not necessarily distinct. If any element of S is removed, then the remaining integers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal.

13. Find a number N such that whenever a complete graph on N vertices is colored red and blue, there must be either four vertices all connected by red edges or four vertices all connected by blue edges.

14. Follow-up: given positive integers k_1 and k_2, prove that there exists N such that whenever a complete graph on N vertices is colored red and blue, there must be either k_1 vertices all connected by red edges or k_2 vertices all connected by blue edges. (This is a version of Ramsey's theorem, proved by Frank P. Ramsey in 1928.)

4 points

15. (2002 A3) Let $n \geq 2$ be an integer and T_n be the number of nonempty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

16. (2010 B3) There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Just for fun: look up Don Zagier's paper "A One-Sentence Proof That Every Prime $p \equiv 1 \pmod 4$ Is a Sum of Two Squares", which uses an elementary parity argument to prove a nontrivial theorem in number theory. (The standard proof of this theorem uses some algebraic properties of the ring of Gaussian integers $a+bi$, where $a, b \in \mathbb{Z}$ and $i = \sqrt{-1}$.) Zagier sweeps a lot of details under the rug in order to fit his proof in one sentence—try verifying a few details for yourself!