Math 191 Homework 12: Harder Putnam problems

Due: Friday, December 1, 2017 (note the unusual due date)

Below are some harder (#3-6) problems from past Putnam exams, arranged by problem number. Solve at least 4 of them; aim for more if your goal on the Putnam is 30+ points. You can also try other Putnam problems #3-6 from 1985 or later, provided that you don't already know how to solve them. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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1. (1985 B3) Let

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

   be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

2. (1990 A3) Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to $5/2$.

3. (1991 B3) Does there exist a real number L such that, if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

4. (2004 A3) Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that $\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$ for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

5. (2006 A3) Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

6. (2014 A3) Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute $\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$ in closed form.

7. (2015 A3) Compute $\log_2 \left(\prod_{a=1}^{2015} \prod_{b=1}^{2015} (1 + e^{2\pi i ab/2015}) \right)$. Here i is the imaginary unit (that is, $i^2 = -1$).

8. (1989 B4) Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

9. (1990 A4) Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

10. (1991 A4) Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that

    (a) the c_i have no limit point in the finite plane,

    (b) the sum of the areas of the D_i is finite, and

    (c) every line in the plane intersects at least one of the D_i?

11. (2004 B4) Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots$, then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

12. (2005 B4) For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

13. (2008 A4) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln x) & \text{if } x > e. \end{cases}$$

    Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

14. (2015 B4) Let T be the set of all triples (a, b, c) of positive integers for which there exist triangles with side lengths a, b, c. Express

$$\sum_{(a,b,c) \in T} \frac{2^a}{3^b 5^c}$$

    as a rational number in lowest terms.

15. (1988 A5) Prove that there exists a *unique* function f from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that $f(f(x)) = 6x - f(x)$ and $f(x) > 0$ for all $x > 0$.

16. (1997 A5) Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

17. (1988 A6) If a linear transformation A on an n-dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.