Math 191 Homework 11: Easier Putnam problems

Due: Monday, November 20, 2017

Below are some relatively easy (#1-2) problems from recent Putnam exams, in chronological order. Solve at least 8 of them; aim for more if your goal on the Putnam is more than 20 points. You can also try other Putnam problems from 1985 or later, provided that you don't already know how to solve them. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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1. (1968 A1) Prove that $\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$.

2. (1988 B2) Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

3. (1992 B1) Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?

4. (1994 A1) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

5. (2000 A2) Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2+0^2, 1 = 0^2+1^2, 2 = 1^2+1^2$.]

6. (2000 B1) Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of $j, 1 \leq j \leq N$.

7. (2001 B1) Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is $(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n$. Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

8. (2005 B1) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)

9. (2008 A1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

10. (2009 A1) Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follows that $f(P) = 0$ for all points P in the plane?

11. (2010 B1) Is there an infinite sequence of real numbers $a_1, a_2, a_3, \dots$ such that $a_1^m + a_2^m + a_3^m + \cdots = m$ for every positive integer m?

12. (2010 B2) Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

13. (2011 A1) Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0,0), P_1, \dots, P_n$ such that $n \geq 2$ and:

    - the directed line segments $P_0P_1, P_1P_2, \dots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.;
    - the lengths of these line segments are positive and strictly increasing.

    [Picture omitted.] How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n, of any growing spiral?

14. (2011 A2) Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \dots$. Assume that the sequence (b_j) is bounded. Prove that $S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$ converges, and evaluate S.

15. (2013 B1) For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of $\sum_{n=1}^{2013} c(n)c(n+2)$.

16. (2013 B2) Let $C = \cup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form $f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi nx)$ for which: (i) $f(x) \geq 0$ for all real x, and (ii) $a_n = 0$ whenever n is a multiple of 3. Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

17. (2014 A1) Prove that every nonzero coefficient of the Taylor series of $(1 - x + x^2)e^x$ about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

18. (2014 B1) A *base 10 over-expansion* of a positive integer N is an expression of the form $N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$ with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \dots, 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

19. (2015 A2) Let $a_0 = 1, a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

20. (2016 A1) Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer $p^{(j)}(k) = \frac{d^j}{dx^j} p(x)|_{x=k}$ (the j-th derivative of $p(x)$ at k) is divisible by 2016.

21. (2016 B1) Let $x_0, x_1, x_2, \dots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$, $x_{n+1} = \ln(e^{x_n} - x_n)$ (as usual, the function ln is the natural logarithm). Show that the infinite series $x_0 + x_1 + x_2 + \cdots$ converges and find its sum.