Math 191 Homework 10: Real analysis

Due: Monday, November 13, 2017

The problems are weighted by (approximate) difficulty. Solve at least 14 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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2 points

1. (1938 #1—the first Putnam problem!) A solid is bounded by two bases in the horizontal planes $z = h/2$ and $z = -h/2$, and by such a surface that the area of every section in a horizontal plane is given by a formula of the sort

$$\text{Area} = a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

(where as special cases some of the coefficients may be 0). Show that the volume is given by the formula $V = \frac{1}{6}h[B_1 + B_2 + 4M]$, where B_1 and B_2 are the areas of the bases, and M is the area of the middle horizontal section. Show that the formulas for the volume of a cone and of a sphere can be included in this formula when $a_0 = 0$.

2. (2006 B5) For each continuous function $f : [0,1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\,\mathrm{d}x$ and $J(x) = \int_0^1 x(f(x))^2\,\mathrm{d}x$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

3. (1989 A2) Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\,\mathrm{d}y\,\mathrm{d}x$ where a and b are positive.

4. (1987 B1) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,\mathrm{d}x}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

5. (1988 A2) A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a,b) and a nonzero function g defined on (a,b) such that this wrong product rule is true for x in (a,b).

6. (2014 B2) Suppose that f is a function on the interval $[1,3]$ such that $-1 \le f(x) \le 1$ for all x and $\int_1^3 f(x)\,\mathrm{d}x = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,\mathrm{d}x$ be?

7. Given two real numbers $a, b > 1$, we say *a overpowers b* if $a^b > b^a$. Find, with proof, a number in $(1,\infty)$ that overpowers all others.

8. (1990 A2) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ ($n, m = 0, 1, 2, \ldots$)?

3 points

9. (Alexander Givental, 2016 mock Putnam #3) Given a sequence $\{a_n\}$ of real numbers such that

$$\lim_{n\to\infty}\left(a_{n+1}-\frac{a_n}{2}\right)=0,$$

prove that $\lim_{n\to\infty} a_n = 0$.

10. (2010 A2) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n)-f(x)}{n}$$

for all real numbers x and all positive integers n.

11. (2007 B2) Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\mathrm{d}x = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left|\int_0^\alpha f(x)\,\mathrm{d}x\right| \le \frac{1}{8}\max_{0\le x\le 1}|f'(x)|.$$

12. (1988 B4) Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty}(a_n)^{n/(n+1)}$.

4 points

13. (2010 A3) Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \le M$ for all $(x,y) \in \mathbb{R}^2$, then h is identically zero.

14. (1998 A3) Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that $f(a)\cdot f'(a)\cdot f''(a)\cdot f'''(a) \ge 0$.

5 points

15. (2010 B5) Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

16. (2012 A6) Let $f(x,y)$ be a continuous, real-valued function on $\mathbb{R}^2$. Suppose that, for every rectangular region R of area 1, the double integral of $f(x,y)$ over R equals 0. Must $f(x,y)$ be identically 0?