Math 191 Homework 1: Induction

Due: Friday, September 1, 2017

The problems are weighted by (approximate) difficulty. Solve at least 12 points worth of problems; don't count problems whose solutions you've seen before. Complete proofs are required for all problems. As always, you must write your solutions up by yourself, and you must cite any ideas that aren't your own.

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1 point

1. Prove the formula $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n > 0$.

2. (1990 A1) Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3} \tag{1}$$

   The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392. \tag{2}$$

   Find with proof a formula for T_n of the form $T_n = A_n + B_n$ where A_n and B_n are well-known sequences.

3. Show that for all $n > 1$, a graph with $2n$ vertices and more than n^2 edges must contain a triangle.

4. Show that every rational number has a finite continued fraction representation

$$a_0 + \cfrac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_k}}}, \tag{3}$$

   where a_0 is an integer and $a_1, \ldots, a_k$ are positive integers.

2 points

5. (2009 B1) Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}. \tag{4}$$

6. Prove that every integer can be expressed as

$$\pm 1^2 \pm 2^2 \pm \cdots \pm n^2, \tag{5}$$

   for some choice of n and some choice of signs.

7. (2002 B1) Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(\frac{x_1+x_2}{2}) = \frac{f(x_1)+f(x_2)}{2}$ for all x_1, x_2. Prove that

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) = \frac{f(x_1) + \cdots + f(x_n)}{n} \tag{6}$$

holds for all n and all x_i.

9. Consider an infinite first-quadrant grid, consisting of the points (a, b) for integers $a, b > 0$. Suppose I choose one of these points (a_1, b_1), and cross out all points (a, b) in the quadrant above and to the right of it; i.e. such that $a \geq a_1$ and $b \geq b_1$. Then I choose another point (a_2, b_2) that is not yet crossed out, and I cross out the quadrant above and to the right of it, and so on. Prove that this process must terminate with all points crossed out in a finite number of steps.

10. Let G be a finite connected graph embedded in the plane, with no edges intersecting except at their endpoints. Prove that $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces, i.e. the number of regions (including the outer region) into which G divides the plane.

4 points

11. (Reid Barton) A deck of 50 cards contains two cards labeled n for each $n = 1, 2, \ldots, 25$. There are 25 people seated at a table, each holding two of the cards in this deck. Each minute every person passes the lower-numbered card of the two they are holding to the right. Prove that eventually someone has two cards of the same number.

12. (2013 B3) Let P be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

    (i) if $S, S' \in P$, then $S \cup S' \in P$ and $S \cap S' \in P$, and

    (ii) if $S \in P$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in P$ and T contains exactly one fewer element than S.

    Suppose that $f : P \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \text{ for all } S, S' \in P. \tag{7}$$

    Must there exist real numbers $f_1, \ldots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i \tag{8}$$

    for every $S \in P$?