Degeneration of the conjugate spectral sequence mod torsion

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Setup

Let X be a smooth proper variety over a perfect field k of characteristic p. Let $W = W(k)$, $\sigma : W \to W$ the Witt vector Frobenius, and $K = \operatorname{Frac} W$.

The de Rham-Witt complex of X/k, first constructed by Illusie in 1979, is designed to lift the de Rham complex $\Omega^*_{X/k}$ to characteristic 0, and thereby to compute crystalline cohomology. It is defined as the initial object in a rather complicated category. Rather than giving its full definition, I will just recall what kinds of structure it has, and some of the key conditions we impose.

The de Rham-Witt complex

It contains the data:

$$\begin{tikzcd}
\vdots \arrow[d, "R"] & \vdots \arrow[d, "R"] & \\
W_2\mathcal{O}_X \arrow[r, "d"] \arrow[d, "R"] & W_2\Omega^1_X \arrow[r, "d"] \arrow[d, "R"] & \cdots \\
W_1\mathcal{O}_X \arrow[r, "d"] & W_1\Omega^1_X \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Here each $W_n\Omega^i_X$ is a sheaf of $W_n\mathcal{O}_X$-modules, with W_nk-linear differentials and vertical quotient maps. (The bottom row is just the de Rham complex of X, and the leftmost column is the sheaf of Witt vectors of $\mathcal{O}_X$.)

Additionally, each row has a multiplicative structure, which I won't use today. Finally, each column has maps F going down and V going up, satisfying the following relations:

(a) $FV = VF = p$,

(b) $dF = pFd$, $Vd = pdV$, $FdV = d$,

(c) $F(a\omega) = \sigma(a)F(\omega)$ and $V(a\omega) = \sigma^{-1}(a)V(\omega)$ for $a \in W$,

and various others.

The complex $W\Omega^*_X$ is defined as $\lim_{\leftarrow} W_n\Omega^*_X$. The F, V, and d operators and the multiplication map pass to the inverse limit, and they have the same relations as above. Given $W\Omega^*_X$ with all of these operators, we can recover $W_n\Omega^*_X$ as its quotient by the images of V^n and dV^n. In practice, we pass between $W\Omega^*_X$ and $(W_n\Omega^*_X)_n$ more or less freely, but one must be somewhat cautious about what operations do and don't commute with the limit.

The de Rham-Witt complex

Remark: Under our smoothness hypotheses, $W\Omega_X^*$ turns out to be p-torsion-free. Then the relations $dF = pFd$, $Vd = pdV$, and $FdV = d$ are all equivalent, and they are equivalent to saying that the σ-semilinear map φ defined by $p^i F$ on $W\Omega^i$ commutes with d. (We will call this the *un-divided Frobenius*.) This is useful because it means the operator φ will pass to various objects in our spectral sequences, including crystalline cohomology, and all of the maps that come up will be compatible with φ.

The most fundamental fact about the de Rham-Witt complex is as follows:

Theorem (Illusie)

The (hyper)cohomology of the de Rham-Witt complex computes crystalline cohomology. More precisely, we have isomorphisms

$$H^*_{\mathrm{cris}}(X/W_n) \cong H^*(W_n\Omega^*_X) := R^*\Gamma(W_n\Omega^*_X)$$
$$H^*_{\mathrm{cris}}(X/W) \cong H^*(W\Omega^*_X) := R^*\Gamma(W\Omega^*_X).$$

Spectral sequence associated to a filtered complex

Given a complex of sheaves K^* equipped with a filtration, there is a spectral sequence allowing us to compute its cohomology in terms of the cohomology of the associated graded objects. There are two natural choices of filtration here, and both give interesting spectral sequences. (I'll discuss the spectral sequences for $W_n\Omega^*_X$; the corresponding statements for $W\Omega^*_X$ follow if we are careful about R^i lim's.)

Slope spectral sequence

The *slope spectral sequence* comes from the *stupid filtration* $\sigma_{\geq i} W_n\Omega^*_X$,

$$\begin{tikzcd}
\sigma_{\geq i} W_n\Omega^*_X \arrow[d, hook] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & W_n\Omega^i_X \arrow[r] \arrow[d, equal] & W_n\Omega^{i+1}_X \arrow[r] \arrow[d, equal] & \cdots \\
W_n\Omega^*_X & \cdots \arrow[r] & W_n\Omega^{i-1}_X \arrow[r] & W_n\Omega^i_X \arrow[r] & W_n\Omega^{i+1}_X \arrow[r] & \cdots
\end{tikzcd}$$

with graded pieces $\mathrm{gr}^i W_n\Omega^*_X = W_n\Omega^i_X[-i]$.

Slope spectral sequence

It has the form:

$$
\begin{aligned}
{}'_nE_1^{i,j} = H^j(W_n\Omega_X^i) &\implies H^{i+j}(W_n\Omega_X^*) = H^{i+j}_{\text{cris}}(X/W_n) \text{ or} \\
{}'E_1^{i,j} = H^j(W\Omega_X^i) &\implies H^{i+j}(W\Omega_X^*) = H^{i+j}_{\text{cris}}(X/W)
\end{aligned}
$$

(*Notation*: we will always use $'E$ to refer to the first spectral sequence and $''E$ for the second. The left subscript n indicates that we are working over W_n instead of W.)

Conjugate spectral sequence

The *conjugate spectral sequence* comes from the *canonical filtration* $\tau_{\leq i} W_n\Omega_X^*$,

$$\begin{tikzcd}
\tau_{\leq i} W_n\Omega_X^* \arrow[d, hook] & \cdots \arrow[r] & W_n\Omega_X^{i-1} \arrow[r] \arrow[d, equal] & \ker(d^i) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
W_n\Omega_X^* & \cdots \arrow[r] & W_n\Omega_X^{i-1} \arrow[r] & W_n\Omega_X^i \arrow[r] & W_n\Omega_X^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

with graded pieces

$$\begin{aligned}
\operatorname{gr}^i W_n\Omega_X^* &= (W_n\Omega_X^{i-1}/\ker(d^{i-1}) \overset{d^{i-1}}{\hookrightarrow} \ker(d^i)) \\
&\overset{\text{q.i.}}{\simeq} \mathscr{H}^i(W_n\Omega_X^*)[-i]
\end{aligned}$$

Conjugate spectral sequence

Here the $''E_1$ page is not canonical, but the $''E_2$ page is:

$$''_nE_2^{ij} = H^i(X, \mathscr{H}^j(W_n\Omega_X^*)) \implies H^{i+j}(W_n\Omega_X^*) = H^{i+j}_{\text{cris}}(X/W_n), \text{ or}$$
$$''E_2^{ij} = H^i(X, \mathscr{H}^j(W\Omega_X^*)) \implies H^{i+j}(W\Omega_X^*) = H^{i+j}_{\text{cris}}(X/W),$$

where $\mathscr{H}^j$ denotes the cohomology sheaves—literally cocycles mod coboundaries.

Spectral sequences

To make this concrete, suppose we have a sufficiently nice (i.e. Cartan-Eilenberg) injective resolution I^{**} of the complex $W_n\Omega^i_X$. This is a double complex of sheaves of W_n-modules of the form

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \iddots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{01} & \longrightarrow & I^{11} & \longrightarrow & I^{21} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{00} & \longrightarrow & I^{10} & \longrightarrow & I^{20} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & W_n\Omega^0_X & \longrightarrow & W_n\Omega^1_X & \longrightarrow & W_n\Omega^2_X & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then $W_n\Omega_X^*$ is quasi-isomorphic to the total complex $\mathrm{Tot}(I^{**})$. In particular, we can compute its cohomology as the cohomology of $\mathrm{Tot}(\Gamma(I^{**}))$. This in turn can be computed by running a spectral sequence whose E_0 page is the following (non-canonical) double complex:

Spectral sequences

E_0:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \iddots \\
0 \arrow[r] & \Gamma(I^{02}) \arrow[r] \arrow[u] & \Gamma(I^{12}) \arrow[r] \arrow[u] & \Gamma(I^{22}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \Gamma(I^{01}) \arrow[r] \arrow[u] & \Gamma(I^{11}) \arrow[r] \arrow[u] & \Gamma(I^{21}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \Gamma(I^{00}) \arrow[r] \arrow[u] & \Gamma(I^{10}) \arrow[r] \arrow[u] & \Gamma(I^{20}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Spectral sequences

But given a double complex, there are two different associated spectral sequences. Starting with vertical maps leads to the slope spectral sequence, and starting with horizontal maps leads to the conjugate spectral sequence.

Another interpretation of the conjugate spectral sequence

I lied earlier: the most fundamental fact about about $W\Omega^*_X$ is *not* that its hypercohomology computes crystalline cohomology. Rather, the global sections functor $\Gamma : (X/W)_{\text{cris}} \to \text{Sh}(*)$ factors through $\text{Sh}(X_{\text{Zar}})$:

$$(X/W)_{\text{cris}} \xrightarrow{u_*} \text{Sh}(X_{\text{Zar}}) \xrightarrow{\Gamma} \text{Sh}(*)$$

and therefore the cohomology functor $R\Gamma : D((X/W)_{\text{cris}}) \to D(\text{Sh}(*))$ factors through $D(\text{Sh}(X_{\text{Zar}}))$:

$$D((X/W)_{\text{cris}}) \xrightarrow{Ru_*} D(\text{Sh}(X_{\text{Zar}})) \xrightarrow{R\Gamma} D(\text{Sh}(*)).$$

Another interpretation of the conjugate spectral sequence

Crystalline cohomology is defined as $R\Gamma(\mathcal{O}_{X/W})$. The most fundamental fact about the de Rham-Witt complex is that it's a representative of the derived Zariski sheaf $Ru_*(\mathcal{O}_{X/W})$ as an honest complex of sheaves.

It follows from this that $R\Gamma(W\Omega^*_X)$ equals crystalline cohomology, as $R\Gamma \circ Ru_* = R\Gamma$. The conjugate spectral sequence appears here as the Leray spectral sequence for the composition of two derived functors. In particular, this shows that the conjugate spectral sequence is interesting to study even if a priori we are only interested in crystalline cohomology and not in the de Rham-Witt complex.

Spectral sequences

Let's compare our two spectral sequences (over W):

$$'E_1^{i,j} = H^j(W\Omega_X^i) \implies H_{\mathrm{cris}}^{i+j}(X/W),$$
$$''E_2^{i,j} = H^i(X, \mathscr{H}^j(W\Omega_X^*)) \implies H_{\mathrm{cris}}^{i+j}(X/W)$$

Note that there are three differences:

- The first starts at E_1 and the second at E_2.
- The roles of i and j get switched (because of starting with horizontal vs. vertical maps).
- The first involves the sheaf cohomology of $W\Omega_X^i$ itself, whereas the second involves the sheaf cohomology of the cohomology sheaves of the complex $W\Omega_X^*$.

Example

For a typical example of what the two spectral sequences look like, let X/k be a supersingular abelian surface. Then the ${}'E_1$ page of the slope spectral sequence looks like:

$$\begin{tikzcd}
k[[x]] \arrow[r, hook] & k[[x]] \oplus W^{\oplus 4} & W \\
W^{\oplus 4} & W^{\oplus 6} & 0 \\
W & 0 & 0
\end{tikzcd}$$

Example

The $''E_2$ page of the spectral sequence looks (I think) like:

$$\begin{tikzcd}
0 & k[[x]] & W \\
0 & W^{\oplus 6} & W^{\oplus 4} \\
W & W^{\oplus 4} & k[[x]] \arrow[uul, "\simeq"']
\end{tikzcd}$$

In both cases, all maps are zero except for the indicated maps on torsion, and the spectral sequences degenerate on the following page with no torsion.

Proposition (Illusie)

For all i, j, and n, the W_n-module $H^j(W_n\Omega^i_X)$ has finite length.

Remark: This is needed to prove that $H^j(W\Omega^i_X) = \lim_{\leftarrow n} H^j(W_n\Omega^i_X)$, so that the slope spectral sequence over W is the inverse limit of the ones over W_n.

Note that in our example, some $H^j(W_n\Omega^i_X)$ have infinitely much p-torsion, but only finitely much of this appears over any given W_n.

Slopes

Before stating Illusie's main result, let me briefly explain the "slope" terminology. The un-divided Frobenius φ mentioned earlier induces operators φ on each $H^j(W\Omega^i)$, and therefore on $H^j(W\Omega^i)/\,\text{tors}$. These are Frobenius-semilinear maps on finite free W-modules. Any such object has a collection of *slopes*, which are the semilinear analogues of p-adic valuations of eigenvalues. Since the divided Frobenius F satisfies $FV = p$, with V topologically nilpotent, it must have all its slopes in $[0, 1)$. It follows that $\varphi = p^i F$ on $H^j(W\Omega^i_X)$ has slopes in $[i, i+1)$.

Illusie's degeneration theorem

This discussion implies the theorem:

Theorem (Illusie)

The slope spectral sequence degenerates at ${}'E_1$ mod torsion (i.e. after applying $\otimes_W K$), and the i-th graded piece of the induced filtration is the part of $H^*_{\text{cris}}(X/W) \otimes_W K$ with slope in $[i, i+1)$.

Proof: All ${}'E_1^{i,j}$ have φ operators with slopes in $[i, i+1)$, and all differentials respect φ. It follows that the ${}'E_n^{i,j}$ for $n \geq 1$ inherit φ, also with slopes in $[i, i+1)$, and also commuting with differentials. But the differentials on page ${}'E_1$ and beyond go between modules with no slopes in common, so they're 0 mod torsion.

Results of Illusie-Raynaud

Lemma (Illusie-Raynaud)

For all i, j, and n, the W_n-module $H^j(\mathscr{H}^i(W_n\Omega_X^*))$ has finite length.

Proof: For each n, we have a so-called *higher Cartier isomorphism*

$$C^{-n} : W_n\Omega_X^i \xrightarrow{\sim} \mathscr{H}^i(W_n\Omega_X^*),$$

which is a σ^n-semilinear isomorphism of sheaves of W_n-modules on X. (In the case $n = 1$, this is the usual Cartier isomorphism.) Taking cohomology on both sides gives us σ^n-semilinear isomorphisms

$$'_nE_1^{ij} = H^j(X, W_n\Omega_X^i) \cong H^j(X, \mathscr{H}^i(W_n\Omega_X^*)) = {}''_nE_2^{ji}.$$

Since Illusie showed the left side has finite length, the right side does too.

Warning: These isomorphisms are *not* compatible as n varies, so we cannot get an isomorphism of objects over W by passing to the limit.

Illusie-Raynaud's degeneration theorem

Main theorem (Illusie-Raynaud)

The conjugate spectral sequence degenerates at $''E_2$ mod torsion (i.e. after applying $\otimes_W K$), and the j-th graded piece of the induced filtration is the part of $H^*(X/W) \otimes_W K$ with slope in $(j-1, j]$.

Sketch of proof

Recall that in Illusie's proof of degeneration, the key idea was as follows. The object ${}'E_1^{ij} = H^j(W\Omega^i_X)$ comes with F and V operators, such that $\varphi = p^i F$ is compatible with the maps in the spectral sequence and has slopes in $[i, i+1)$. Since these intervals are disjoint for different i, it followed that all subsequent maps in the spectral sequence (mod torsion) vanish.

Sketch of proof

We want to imitate this for $''E_2^{ij} = H^i(\mathscr{H}^j(W\Omega_X^*))$. The problem is that neither F nor V induces a well-defined operator on $\mathscr{H}^j(W\Omega_X^*)$: F preserves the cocycles $ZW\Omega_X^i$ but not the coboundaries $BW\Omega_X^i$, and V preserves coboundaries but not cocycles. Instead we define $F' = pF : W\Omega^i \to W\Omega^i$ and $V' = F^{-1}|_{ZW\Omega_X^i}$. These maps preserve both cocycles and coboundaries, so they induce maps on $\mathscr{H}^j(W\Omega_X^*)$ and thus $H^i(\mathscr{H}^j(W\Omega_X^*))$. These have the right semilinearity properties, and they satisfy $F'V' = V'F' = p$.

Sketch of proof

The operator $F' = pF$ is topologically nilpotent on $W\Omega^i$, since p is. It follows that F' is topologically nilpotent (albeit no longer divisible by p) as an operator on $\mathscr{H}^j(W\Omega^*_X)$, and also on $H^i(\mathscr{H}^j(W\Omega^*_X))/\operatorname{tors}$. Then the slopes of F' are in $(0,1]$, so the slopes of $\varphi = p^j F = p^{j-1}F'$ are in $(j-1, j]$. From here the proof concludes as in Illusie.

Sketch of proof

So what's the catch? This is a 140-page paper, right?

Showing that F' and V' give well-defined maps on $\mathscr{H}^j(W\Omega_X^*)$ takes some work, but not that much. But the main issue is that in order to talk about slopes, we need to know that $H^i(\mathscr{H}^j(W\Omega_X^*))/\operatorname{tors}$ is a finite free W-module. Proving this requires a precise understanding of what kind of object $H^i(\mathscr{H}^j(W\Omega_X^*))$ is.

The Raynaud ring

Let R denote the noncommutative graded W-algebra generated by elements F and V in degree 0 and d in degree 1, subject to all the usual relations:

- $FV = VF = p$, $d^2 = 0$,
- $dF = pFd$, $Vd = pdV$, $FdV = d$,
- $F(a\omega) = \sigma(a)F(\omega)$ and $V(a\omega) = \sigma^{-1}(a)V(\omega)$ for $a \in W$.

This ring is concentrated in degrees 0 and 1. It is a free W-module with basis:

$$\{F^m, V^n, F^m d, dV^n : m \geq 0, n > 0\}.$$

Any complex of W-modules with suitable F and V operators is then a (graded left) R-module. Given such a module M^*, we define

$$W_n M^i = M^i / (V^n M^i, dV^n M^{i-1}).$$

Structure theorem for (nice) R-modules

Proposition (Illusie-Raynaud)

Suppose M^* is a graded left R-module, concentrated in finitely many degrees, such that $M^* = \lim_{\leftarrow n} W_n M^*$ and each $W_n M^i$ is a finite-length W-module. Then M^* has a finite filtration with quotients of the following types:

1. Concentrated in one degree:
   - (a) finite-length torsion W-modules,
   - (b) finite free W-modules,
   - (c) $k[[V]]$, with V acting semilinearly on k and $F = 0$,
2. "dominoes," denoted $U_i[-n]$, concentrated in degrees n and $n+1$.

In particular, each ${}''E_2^{ij} = H^i(\mathscr{H}^j(W\Omega_X^*))$ satisfies these hypotheses, so one can use the proposition to prove finiteness properties about it.

Applications and related work

- Theorem (Rudakov-Shafarevich): A K3 surface X over an arbitrary field k has no global vector fields. This is easy in characteristic 0. In the characteristic-p case, they use Illusie-Raynaud's theory of dominoes to study various differentials in the Hodge-de Rham, slope, and conjugate spectral sequences, and eventually show that $H^0(X, T_X) \cong H^0(X, \Omega^1_X) = 0$.
- Ekedahl's thesis uses some further study of the category of R-modules to show how one can prove Künneth and duality formulas for crystalline cohomology using the de Rham-Witt complex.
- Katz ("Crystalline cohomology, Dieudonné modules, and Jacobi sums", 1981) gives a formula for Gauss sums using the degeneration of the conjugate spectral sequence for Artin-Schreier covers of $\mathbb{P}^1$.