Degeneration of the conjugate spectral sequence mod torsion

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Setup

Let X be a smooth proper variety over a perfect field k of characteristic p. Let $W = W(k)$, $\sigma : W \to W$ the Witt vector Frobenius, and $K = \operatorname{Frac} W$.

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The de Rham-Witt complex of X/k, first constructed by Illusie in 1979, is designed to lift the de Rham complex $\Omega^*_{X/k}$ to characteristic 0, and thereby to compute crystalline cohomology. It is defined as the initial object in a rather complicated category. Rather than giving its full definition, I will just recall what kinds of structure it has, and some of the key conditions we impose.

The de Rham-Witt complex

It contains the data:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\Big\downarrow{\scriptstyle R} & & \Big\downarrow{\scriptstyle R} & & \\
W_2\mathcal{O}_X & \xrightarrow{\ d\ } & W_2\Omega^1_X & \xrightarrow{\ d\ } & \cdots \\
\Big\downarrow{\scriptstyle R} & & \Big\downarrow{\scriptstyle R} & & \\
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Here each $W_n\Omega^i_X$ is a sheaf of $W_n\mathcal{O}_X$-modules, with W_nk-linear differentials and vertical quotient maps.

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Here each $W_n\Omega^i_X$ is a sheaf of $W_n\mathcal{O}_X$-modules, with W_nk-linear differentials and vertical quotient maps. (The bottom row is just the de Rham complex of X, and the leftmost column is the sheaf of Witt vectors of $\mathcal{O}_X$.)

The de Rham-Witt complex

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(a) $FV = VF = p$,

(b) $dF = pFd$, $Vd = pdV$, $FdV = d$,

(c) $F(a\omega) = \sigma(a)F(\omega)$ and $V(a\omega) = \sigma^{-1}(a)V(\omega)$ for $a \in W$,

and various others.

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The de Rham-Witt complex and crystalline cohomology

The most fundamental fact about the de Rham-Witt complex is as follows:

Theorem (Illusie)

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The (hyper)cohomology of the de Rham-Witt complex computes crystalline cohomology. More precisely, we have isomorphisms

$$H^*_{\mathrm{cris}}(X/W_n) \cong H^*(W_n\Omega^*_X) := R^*\Gamma(W_n\Omega^*_X)$$
$$H^*_{\mathrm{cris}}(X/W) \cong H^*(W\Omega^*_X) := R^*\Gamma(W\Omega^*_X).$$

Spectral sequence associated to a filtered complex

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Slope spectral sequence

The *slope spectral sequence* comes from the *stupid filtration* $\sigma_{\geq i} W_n \Omega_X^*$,

$$
\begin{array}{ccccccccccc}
\sigma_{\geq i} W_n \Omega_X^* & & \cdots & \longrightarrow & 0 & \longrightarrow & W_n \Omega_X^i & \longrightarrow & W_n \Omega_X^{i+1} & \longrightarrow & \cdots \\
\big\downarrow & & & & \big\downarrow & & \| & & \| & & \\
W_n \Omega_X^* & & \cdots & \longrightarrow & W_n \Omega_X^{i-1} & \longrightarrow & W_n \Omega_X^i & \longrightarrow & W_n \Omega_X^{i+1} & \longrightarrow & \cdots
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\end{array}
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with graded pieces $\mathrm{gr}^i W_n\Omega_X^* = W_n\Omega_X^i[-i]$.

Slope spectral sequence

It has the form:

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\begin{aligned}
{}_n'E_1^{i,j} = H^j(W_n\Omega_X^i) &\implies H^{i+j}(W_n\Omega_X^*) = H^{i+j}_{\mathrm{cris}}(X/W_n) \text{ or} \\
{}'E_1^{i,j} = H^j(W\Omega_X^i) &\implies H^{i+j}(W\Omega_X^*) = H^{i+j}_{\mathrm{cris}}(X/W)
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\end{aligned}
$$

(*Notation*: we will always use ${}'E$ to refer to the first spectral sequence and ${}''E$ for the second. The left subscript n indicates that we are working over W_n instead of W.)

Conjugate spectral sequence

The *conjugate spectral sequence* comes from the *canonical filtration* $\tau_{\leq i} W_n\Omega_X^*$,

$$
\begin{array}{ccccccccccc}
\tau_{\leq i} W_n\Omega_X^* & & \cdots & \longrightarrow & W_n\Omega_X^{i-1} & \longrightarrow & \ker(d^i) & \longrightarrow & 0 & \longrightarrow & \cdots \\
\big\downarrow & & & & \big\| & & \big\downarrow & & \big\downarrow & & \\
W_n\Omega_X^* & & \cdots & \longrightarrow & W_n\Omega_X^{i-1} & \longrightarrow & W_n\Omega_X^{i} & \longrightarrow & W_n\Omega_X^{i+1} & \longrightarrow & \cdots
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W_n\Omega^*_X & & \cdots \longrightarrow & W_n\Omega^{i-1}_X & \longrightarrow & W_n\Omega^i_X & \longrightarrow & W_n\Omega^{i+1}_X & \longrightarrow \cdots
\end{array}$$

with graded pieces

$$\begin{aligned}
\mathrm{gr}^i\, W_n\Omega^*_X &= (W_n\Omega^{i-1}_X / \ker(d^{i-1}) \overset{d^{i-1}}{\hookrightarrow} \ker(d^i)) \\
&\overset{\text{q.i.}}{\simeq} \mathscr{H}^i(W_n\Omega^*_X)[-i]
\end{aligned}$$

Conjugate spectral sequence

Here the $''E_1$ page is not canonical, but the $''E_2$ page is:

$$''_nE_2^{ij} = H^i(X, \mathscr{H}^j(W_n\Omega_X^*)) \implies H^{i+j}(W_n\Omega_X^*) = H^{i+j}_{\mathrm{cris}}(X/W_n), \text{ or}$$
$$''E_2^{ij} = H^i(X, \mathscr{H}^j(W\Omega_X^*)) \implies H^{i+j}(W\Omega_X^*) = H^{i+j}_{\mathrm{cris}}(X/W),$$

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where $\mathscr{H}^j$ denotes the cohomology sheaves—literally cocycles mod coboundaries.

Spectral sequences

To make this concrete, suppose we have a sufficiently nice (i.e. Cartan-Eilenberg) injective resolution I^{**} of the complex $W_n\Omega^i_X$.

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To make this concrete, suppose we have a sufficiently nice (i.e. Cartan-Eilenberg) injective resolution I^{**} of the complex $W_n\Omega^i_X$. This is a double complex of sheaves of W_n-modules of the form

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \iddots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{01} & \longrightarrow & I^{11} & \longrightarrow & I^{21} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{00} & \longrightarrow & I^{10} & \longrightarrow & I^{20} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & W_n\Omega^0_X & \longrightarrow & W_n\Omega^1_X & \longrightarrow & W_n\Omega^2_X & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Then $W_n\Omega^*_X$ is quasi-isomorphic to the total complex $\mathrm{Tot}(I^{**})$. In particular, we can compute its cohomology as the cohomology of $\mathrm{Tot}(\Gamma(I^{**}))$.

Then $W_n\Omega^*_X$ is quasi-isomorphic to the total complex $\mathrm{Tot}(I^{**})$. In particular, we can compute its cohomology as the cohomology of $\mathrm{Tot}(\Gamma(I^{**}))$. This in turn can be computed by running a spectral sequence whose E_0 page is the following (non-canonical) double complex:

Spectral sequences

E_0:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \iddots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I^{02}) & \longrightarrow & \Gamma(I^{12}) & \longrightarrow & \Gamma(I^{22}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I^{01}) & \longrightarrow & \Gamma(I^{11}) & \longrightarrow & \Gamma(I^{21}) & \longrightarrow & \cdots \\
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0 & \longrightarrow & \Gamma(I^{00}) & \longrightarrow & \Gamma(I^{10}) & \longrightarrow & \Gamma(I^{20}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

But given a double complex, there are two different associated spectral sequences. Starting with vertical maps leads to the slope spectral sequence, and starting with horizontal maps leads to the conjugate spectral sequence.

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$$(X/W)_{\mathrm{cris}} \xrightarrow{u_*} \mathrm{Sh}(X_{\mathrm{Zar}}) \xrightarrow{\Gamma} \mathrm{Sh}(*)$$

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$$(X/W)_{\text{cris}} \xrightarrow{u_*} \text{Sh}(X_{\text{Zar}}) \xrightarrow{\Gamma} \text{Sh}(*)$$

and therefore the cohomology functor $R\Gamma : D((X/W)_{\text{cris}}) \to D(\text{Sh}(*))$ factors through $D(\text{Sh}(X_{\text{Zar}}))$:

$$D((X/W)_{\text{cris}}) \xrightarrow{Ru_*} D(\text{Sh}(X_{\text{Zar}})) \xrightarrow{R\Gamma} D(\text{Sh}(*)).$$

Another interpretation of the conjugate spectral sequence

Crystalline cohomology is defined as $R\Gamma(\mathcal{O}_{X/W})$. The most fundamental fact about the de Rham-Witt complex is that it's a representative of the derived Zariski sheaf $Ru_*(\mathcal{O}_{X/W})$ as an honest complex of sheaves.

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It follows from this that $R\Gamma(W\Omega_X^*)$ equals crystalline cohomology, as $R\Gamma \circ Ru_* = R\Gamma$. The conjugate spectral sequence appears here as the Leray spectral sequence for the composition of two derived functors. In particular, this shows that the conjugate spectral sequence is interesting to study even if a priori we are only interested in crystalline cohomology and not in the de Rham-Witt complex.

Spectral sequences

Let's compare our two spectral sequences (over W):

$$
\begin{aligned}
{}'E_1^{i,j} &= H^j(W\Omega_X^i) \implies H_{\text{cris}}^{i+j}(X/W), \\
{}''E_2^{i,j} &= H^i(X, \mathscr{H}^j(W\Omega_X^*)) \implies H_{\text{cris}}^{i+j}(X/W)
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Note that there are three differences:

- The first starts at E_1 and the second at E_2.
- The roles of i and j get switched (because of starting with horizontal vs. vertical maps).
- The first involves the sheaf cohomology of $W\Omega_X^i$ itself, whereas the second involves the sheaf cohomology of the cohomology sheaves of the complex $W\Omega_X^*$.

Example

For a typical example of what the two spectral sequences look like, let X/k be a supersingular abelian surface. Then the ${}'E_1$ page of the slope spectral sequence looks like:

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$$\begin{array}{ccc}
k[[x]] \hookrightarrow & k[[x]] \oplus W^{\oplus 4} & W \\
W^{\oplus 4} & W^{\oplus 6} & 0 \\
W & 0 & 0
\end{array}$$

Example

The ${}^{\prime\prime}E_2$ page of the spectral sequence looks (I think) like:

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The $''E_2$ page of the spectral sequence looks (I think) like:

$$\begin{array}{ccc} 0 & k[[x]] & W \\ 0 & W^{\oplus 6} \quad \simeq & W^{\oplus 4} \\ W & W^{\oplus 4} & k[[x]] \end{array}$$

Example

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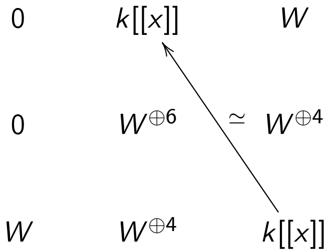

In both cases, all maps are zero except for the indicated maps on torsion, and the spectral sequences degenerate on the following page with no torsion.

Results of Illusie

Proposition (Illusie)

For all i, j, and n, the W_n-module $H^j(W_n\Omega^i_X)$ has finite length.

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Note that in our example, some $H^j(W_n\Omega^i_X)$ have infinitely much p-torsion, but only finitely much of this appears over any given W_n.

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Illusie's degeneration theorem

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The slope spectral sequence degenerates at ${}'E_1$ mod torsion (i.e. after applying $\otimes_W K$), and the i-th graded piece of the induced filtration is the part of $H^*_{\mathrm{cris}}(X/W) \otimes_W K$ with slope in $[i, i+1)$.

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Proof: All ${}'E_1^{i,j}$ have φ operators with slopes in $[i, i+1)$, and all differentials respect φ.

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Proof: All $'E_1^{i,j}$ have φ operators with slopes in $[i, i+1)$, and all differentials respect φ. It follows that the $'E_n^{i,j}$ for $n \geq 1$ inherit φ, also with slopes in $[i, i+1)$, and also commuting with differentials.

Illusie's degeneration theorem

This discussion implies the theorem:

Theorem (Illusie)

The slope spectral sequence degenerates at ${}'E_1$ mod torsion (i.e. after applying $\otimes_W K$), and the i-th graded piece of the induced filtration is the part of $H^*_{\text{cris}}(X/W) \otimes_W K$ with slope in $[i, i+1)$.

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Results of Illusie-Raynaud

Lemma (Illusie-Raynaud)

For all i, j, and n, the W_n-module $H^j(\mathscr{H}^i(W_n\Omega^*_X))$ has finite length.

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$$C^{-n} : W_n\Omega^i_X \xrightarrow{\sim} \mathscr{H}^i(W_n\Omega^*_X),$$

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Warning: These isomorphisms are *not* compatible as n varies, so we cannot get an isomorphism of objects over W by passing to the limit.

Illusie-Raynaud's degeneration theorem

Main theorem (Illusie-Raynaud)

The conjugate spectral sequence degenerates at $''E_2$ mod torsion (i.e. after applying $\otimes_W K$), and the j-th graded piece of the induced filtration is the part of $H^*(X/W) \otimes_W K$ with slope in $(j-1, j]$.

Sketch of proof

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Showing that F' and V' give well-defined maps on $\mathscr{H}^j(W\Omega^*_X)$ takes some work, but not that much. But the main issue is that in order to talk about slopes, we need to know that $H^i(\mathscr{H}^j(W\Omega^*_X))/\operatorname{tors}$ is a finite free W-module.

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Showing that F' and V' give well-defined maps on $\mathscr{H}^j(W\Omega^*_X)$ takes some work, but not that much. But the main issue is that in order to talk about slopes, we need to know that $H^i(\mathscr{H}^j(W\Omega^*_X))/\operatorname{tors}$ is a finite free W-module. Proving this requires a precise understanding of what kind of object $H^i(\mathscr{H}^j(W\Omega^*_X))$ is.

The Raynaud ring

Let R denote the noncommutative graded W-algebra generated by elements F and V in degree 0 and d in degree 1, subject to all the usual relations:

- $FV = VF = p$, $d^2 = 0$,
- $dF = pFd$, $Vd = pdV$, $FdV = d$,
- $F(a\omega) = \sigma(a)F(\omega)$ and $V(a\omega) = \sigma^{-1}(a)V(\omega)$ for $a \in W$.

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$$\{F^m, V^n, F^m d, dV^n : m \geq 0, n > 0\}.$$

Any complex of W-modules with suitable F and V operators is then a (graded left) R-module. Given such a module M^*, we define

$$W_n M^i = M^i / (V^n M^i, dV^n M^{i-1}).$$

Structure theorem for (nice) R-modules

Proposition (Illusie-Raynaud)

Suppose M^* is a graded left R-module, concentrated in finitely many degrees, such that $M^* = \lim_{\leftarrow n} W_n M^*$ and each $W_n M^i$ is a finite-length W-module.

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1. Concentrated in one degree:
   (a) finite-length torsion W-modules,
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   (c) $k[[V]]$, with V acting semilinearly on k and $F = 0$,
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In particular, each $''E_2^{ij} = H^i(\mathscr{H}^j(W\Omega_X^*))$ satisfies these hypotheses, so one can use the proposition to prove finiteness properties about it.

Applications and related work

- Theorem (Rudakov-Shafarevich): A K3 surface X over an arbitrary field k has no global vector fields. This is easy in characteristic 0. In the characteristic-p case, they use Illusie-Raynaud's theory of dominoes to study various differentials in the Hodge-de Rham, slope, and conjugate spectral sequences, and eventually show that $H^0(X, T_X) \cong H^0(X, \Omega^1_X) = 0$.
- Ekedahl's thesis uses some further study of the category of R-modules to show how one can prove Künneth and duality formulas for crystalline cohomology using the de Rham-Witt complex.
- Katz ("Crystalline cohomology, Dieudonné modules, and Jacobi sums", 1981) gives a formula for Gauss sums using the degeneration of the conjugate spectral sequence for Artin-Schreier covers of $\mathbb{P}^1$.