Torsion comparisons in cohomology

Ravi Fernando – fernando@berkeley.edu

April 26, 2017*

1 Setup and motivation

Fix a prime p, a complete discretely valued extension $K/\mathbb{Q}_p$, and a completed algebraic closure $C = \widehat{\overline{K}}$. Then we have rings of integers $\mathcal{O}_K$ and $\mathcal{O}_C$, and so on. Let $\mathfrak{X}$ be a smooth proper formal scheme over $\mathcal{O}_C$; the case of an actual scheme defined over $\mathcal{O}_K$ is interesting enough, but everything works in greater generality (with no mention of K). We write the generic and special fibers of $\mathfrak{X}$ as X and $\mathfrak{X}_k$ respectively.

Remark 1. Kęstutis has generalized some things to the semistable case, where $\mathfrak{X}_k$ is allowed to have simple normal crossings.

For such $\mathfrak{X}$, we can associate three integral p-adic cohomology theories: $H^i_{\text{ét}}(X, \mathbb{Z}_p)$, $H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O}_C)$, and $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$. These are defined in three different ways; they depend on different parts of $\mathfrak{X}$ (generic fiber, full integral model, special fiber); and they are valued over three different mixed-characteristic valuation rings.

There are various comparison theorems between these cohomology theories after tensoring up to some p-adic period rings, but classically this involves inverting p. For example, the crystalline comparison isomorphism

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{crys}} \cong H^i_{\text{crys}}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} B_{\text{crys}} \tag{1}$$

lives over the $\mathbb{Q}_p$-algebra B_{crys}. And this is for good reason: although both sides are equipped with integral lattices, the isomorphism doesn't generally identify those integral lattices with each other.

Classically, the only way we can get isomorphisms at the integral level is with hypotheses restricting the ramification and/or the cohomological degree. The following theorem of Caruso is a typical result of this type:

*Notes for a talk given in Berkeley's number theory seminar, organized by Kęstutis Česnavičius, Xinyi Yuan, Sug Woo Shin, and Ken Ribet. Main reference: Bhatt-Morrow-Scholze, *Integral p-adic Hodge theory.* Other references: Kęstutis Česnavičius, *The A_{inf}-cohomology in the semistable case*; Xavier Caruso, *Conjecture de l'inertie modérée de Serre*, Bhatt-Morrow-Scholze, *Integral p-adic Hodge theory - announcement.*

Theorem 2. *Assume $\mathfrak{X}$ is an actual scheme defined over $\mathcal{O}_K$, and that K has perfect residue field. Let e be the ramification degree of $K/\mathbb{Q}_p$. There exists at least an abstract isomorphism*

$$(H^i(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k))/p^n \cong (H^i(\mathfrak{X}_k/W(k)))/p^n, \tag{2}$$

provided that $ie < p-1$. In particular, the full torsion agrees under this hypothesis, since sufficiently high powers of p act as 0 on the torsion.[1]

(The inequality $ie < p-1$ is necessary because of pathologies that can happen over $\mathbb{Z}_p[\zeta_p]$, or maybe an unramified extension thereof.) So assuming K is mildly ramified and the cohomological degree isn't too big, some things can work integrally. But if $p = 2$, this is completely useless unless you're interested in H^0.

2 Étale versus de Rham and crystalline

The main integral comparison theorem that BMS proves is as follows.

Theorem 3. *For $\mathfrak{X}$ as above, and for all integers $i, n \geq 0$, we have inequalities*

$$\mathrm{length}_{\mathbb{Z}_p}(H^i_{\acute{e}t}(X, \mathbb{Z}_p)_{\mathrm{tors}}/p^n) \leq \mathrm{length}_{W(k)}(H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tors}}/p^n) \tag{3}$$

$$\mathrm{length}_{\mathbb{Z}_p}(H^i_{\acute{e}t}(X, \mathbb{Z}_p)_{\mathrm{tors}}/p^n) \leq \ell_{\mathcal{O}_C}(H^i_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)_{\mathrm{tors}}/p^n), \tag{4}$$

where $\ell_{\mathcal{O}_C}$ is the normalized length with $\ell_{\mathcal{O}_C}(\mathcal{O}_C/p^\alpha) = \alpha$ for any $\alpha \in \mathbb{Q}^{\geq 0}$. In particular, choosing n sufficiently large gives inequalities between the lengths of the full torsion submodules.

A priori, I have two reasons to maybe believe this. First, one motto is that lots of pathologies can happen in characteristic p geometry, especially regarding p-adic cohomology theories, and these pathologies make the cohomology bigger rather than smaller.

For a better motto, which conveys the main idea of the proof, recall that we now have the A_{inf}-cohomology theory, which interpolates between étale, de Rham, and crystalline cohomology. (Draw the picture of $\operatorname{Spec} A_{\mathrm{inf}}$.) Note that étale torsion appears at the bottom point (with residue field $C^\flat$), while de Rham and crystalline torsion appear at the special point (with residue field k). The latter is a specialization of the former, and cohomology dimensions are upper semicontinuous. Moreover, this even works for lengths of each modulo p^n for all n, with length suitably defined for the de Rham case.

Modulo some details about derived vs. ordinary tensor products, the proof in BMS (for étale vs. crystalline; the de Rham case is done similarly in Česnavičius) boils down to the following commutative algebra lemma about a certain type of A_{inf}-modules.

[1] Caruso's original statement is for the derived-mod-p^n versions $H^i_{\mathrm{\acute{e}t}}(X, \mathbb{Z}/p^n\mathbb{Z})$ and $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W_n(k))$, assuming that either $(i+1)e < p-1$, or $n = 1$ and $ie < p-1$. But for universal coefficient theorem reasons, any disagreement in torsion must appear in the mod-p^n picture one degree higher than in the derived-mod-p^n picture.

Lemma 4. *(Corollary 4.15(ii) in BMS.) Let M be a finitely presented A_{inf}-module such that $M[1/p]$ is free over $A_{\mathrm{inf}}[1/p]$. Then for all $n \geq 1$, we have*

$$\mathrm{length}_{W(C^\flat)}((M \otimes_{A_{\mathrm{inf}}} W(C^\flat))/p^n) \leq \mathrm{length}_{W(k)}((M \otimes_{A_{\mathrm{inf}}} W(k))/p^n). \tag{5}$$

BMS shows that all A_{inf}-cohomology groups $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ satisfy the hypothesis; this is part of the definition of a Breuil-Kisin-Fargues module.

3 de Rham versus crystalline

BMS also shows that for all i, $H^i_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ is torsion-free if and only if $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ is. Rough idea of proof: assuming de Rham or crystalline is torsion-free, the A_{inf}-cohomology is free, and then everything is free.

It was discovered at Arizona Winter School this year[2] that this (and a bit more) can be proved by much more elementary means. Recall that the dimensions of $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} \mathrm{Frac}\, W(k)$ and $H^i_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \otimes_{\mathcal{O}_C} C$ agree by classical results of rational p-adic Hodge theory, and that the derived-mod-p versions $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W_1(k) = k) = H^i_{\mathrm{dR}}(\mathfrak{X}_k/k)$ also agree. One might suspect that such equalities on both the generic and special fibers would imply something integrally, and in fact this is the case. Namely, it follows from an easy homological algebra lemma that $H^i_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ and $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ have the same number of torsion summands. The case of zero torsion summands recovers the remark from BMS.

Lemma 5. *Let R be a rank-1 valuation domain with maximal ideal m, not necessarily discretely valued, and let $K = \mathrm{Frac}\, R$ and $k = R/\mathfrak{m}$. Let $D^\bullet$ be a perfect complex of R-modules. Then the numbers*

$$\dim_K H^i(D^\bullet \otimes_R K) \text{ and} \tag{6}$$
$$\dim_k H^i(D^\bullet \otimes^{\mathbb{L}}_R k), \tag{7}$$

for all i, determine the numbers $\dim_k(H^i(D^\bullet) \otimes_R k)$ for all i, without dependence on the ring R.

Proof. Since $D^\bullet$ is a perfect complex, we can and do write it up to quasi-isomorphism as a bounded complex of finite projective, thus free, modules. From the universal coefficient theorem, we have a noncanonical isomorphism

$$H^i(D^\bullet \otimes_R k) \cong (H^i(D^\bullet) \otimes_R k) \oplus \mathrm{Tor}_1^R(H^{i+1}(D^\bullet), k). \tag{8}$$

Each integral cohomology group is finitely presented, so we can choose an isomorphism

$$H^{i+1}(D^\bullet) \cong R^{\oplus a} \oplus \bigoplus_{j=1}^{b} (R/\alpha_j), \tag{9}$$

[2] Jesse Silliman and Matthew Morrow made the observation independently. Jesse's proof was in the setting of a DVR, and I reworked it into the form presented here.

for some $0 \neq \alpha_j \in \mathfrak{m}$. This gives us an obvious two-step free resolution of $H^{i+1}(D^\bullet)$, from which we compute

$$\begin{aligned} \operatorname{Tor}_1^R(H^{i+1}(D^\bullet), k) &= \ker(k^{\oplus b} \xrightarrow{0} k^{\oplus a+b}) && (10)\\ &= k^{\oplus b}. && (11) \end{aligned}$$

So we have

$$\begin{aligned} \dim_k \operatorname{Tor}_1^R(H^{i+1}(D^\bullet), k) &= b = (a+b) - a && (12)\\ &= \dim_k(H^{i+1}(D^\bullet) \otimes_R k) - \dim_K(H^{i+1}(D^\bullet \otimes_R K)). && (13) \end{aligned}$$

Finally, we can assemble all the dimensions into the equation:

$$\begin{aligned} \dim_k(H^i(D^\bullet \otimes k)) &= \dim_k(H^i(D^\bullet) \otimes k) + \dim_k(\operatorname{Tor}_1^R(H^{i+1}(D^\bullet), k)) && (14)\\ &= \dim_k(H^i(D^\bullet) \otimes k) + \dim_k(H^{i+1}(D^\bullet) \otimes_R k) - \dim_K(H^{i+1}(D^\bullet \otimes_R K)). && (15) \end{aligned}$$

We are given the first and last terms of this, and the middle two terms are the degree-i and degree-$(i+1)$ parts of what we want. But since $D^\bullet$ is a bounded complex, these numbers are zero for sufficiently small or sufficiently large i, so the sums of consecutive numbers determine the individual numbers. □

To see that the lemma implies the comparison, choose $D_1^\bullet$ and $D_2^\bullet$ to be the complexes computing de Rham and crystalline cohomology respectively. By the classical comparisons, their dimensions agree when applying derived base change to either of the two fibers. So the two values of $\dim_k H^i(D^\bullet) \otimes_R k$ must agree. But this is just the number of cyclic summands of the integral cohomology, including torsion and non-torsion; we know there are equally many of the latter and therefore of the former.

4 BMS's first example

BMS also constructed two examples that illustrate the sharpness of their results. I'll briefly discuss the first one, and then look at the second in some more detail.

The first example will be a smooth projective surface $X/\mathbb{Z}_2$ with $H^*_{\text{ét}}(X, \mathbb{Z}_2)$ torsion-free in all degrees, but $H^2_{\text{crys}}(X_{\mathbb{F}_2}/\mathbb{Z}_2)_{\text{tors}} \cong \mathbb{F}_2$. This implies that the "de Rham torsion-free iff crystalline torsion-free" statement cannot extend to étale.

First let $S/\mathbb{Z}_2$ be a smooth "singular" Enriques surface. Here, "singular" means that the group scheme $\operatorname{Pic}^\tau(S/\mathbb{Z}_2) = (\operatorname{Pic}(S/Z_2)/\operatorname{Pic}^0(S/\mathbb{Z}_2))_{\text{tors}}$ is μ_2 rather than the constant group scheme $\mathbb{Z}/2\mathbb{Z}$. Such a surface can be obtained by taking the quotient of a K3 surface $\widetilde{S}$ by a free action of the constant group scheme $\mathbb{Z}/2\mathbb{Z}$. (Aside: over a field of characteristic $\neq 2$, all Enriques surfaces are quotients of K3's by $\mu_2 = \mathbb{Z}/2\mathbb{Z}$. In characteristic 2, the "classical" ones are quotients of some surfaces by μ_2 and have $\operatorname{Pic}^\tau = \mathbb{Z}/2\mathbb{Z}$; the "singular" ones are the other way around; and the "supersingular" ones are quotients by α_2 and have $\operatorname{Pic}^\tau = \alpha_2$.)

It turns out that both the étale and crystalline cohomology of S itself have torsion (just $\mathbb{F}_2$, I think) in degree 2. But we can use S to construct our example, as follows. Let $E/\mathbb{Z}_2$ be any ordinary elliptic curve. Then one can construct an E-torsor $D \to S$ that "contains" the double cover $\widetilde{S}$. This unwinds the torsion in étale cohomology, but leaves the crystalline torsion untouched because $D_k = S_k \times E_k$. Finally, a sufficiently general surface $X \hookrightarrow D$ will have both étale and crystalline H^2 isomorphic to those of D, which gives our counterexample.

Remark 6. Note that de Rham cohomology agrees with crystalline here, because X is defined over $W(k)$.

5 BMS's second example and further questions

BMS's second example is a smooth projective relative surface over $\mathcal{O}_C$ whose second étale cohomology has $\mathbb{Z}/p^2\mathbb{Z}$ torsion, but whose second crystalline cohomology has $k \oplus k$ torsion. This obeys the rules about lengths modulo p^n, but it shows that the inequalities cannot be upgraded to an injective or surjective morphism (after base-changing from $\mathbb{Z}_p$ to $W(k)$).

Motivation: the usual way to get torsion (or some other pathologies) in a p-adic or mod-p cohomology theory follows an idea introduced by Serre in 1958. I'll first sketch the idea in roughly the way it first appeared, and then fast-forward to the current setting.

Let $k = \overline{k}$ be a field of characteristic p, and G some p-group; say $\mathbb{Z}/p\mathbb{Z}$. There is a general procedure (which I won't go into detail about here) to construct a smooth complete intersection surface $\widetilde{X}/k$ that admits a free action by G, yielding a smooth quotient X. Since $\widetilde{X}$ is a complete intersection, $H^*(\widetilde{X})$ (for some choice of cohomology theory H^*) can be controlled, and $H^*(X)$ is calculated by a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(G, H^q(\widetilde{X})) \implies H^{p+q}(X) \tag{16}$$

The (trivial) action of G on $H^0(\widetilde{X})$ produces nontrivial higher group cohomology, which translates into pathologies in the higher cohomology (H^1 or H^2, depending on what pathology you're looking for) of X. For example, this approach can be used to create p-torsion in H^2_{crys}, a Hodge to de Rham spectral sequence over $\overline{\mathbb{F}}_p$ that doesn't degenerate at E_1, and so on.

Aside: the "niceness" of the cohomology of the complete intersection $\widetilde{X}$ can be interpreted as $\widetilde{X}$ being cohomologically a "good enough approximation" of a point, and thus X itself being a "good enough approximation" of BG. This is good motivation, because the cohomology of BG is just group cohomology of G.

In our situation, we want a surface over a mixed characteristic valuation ring with interesting torsion in cohomology. We will construct the torsion by quotienting some nice smooth complete intersection by a finite flat group scheme $G/\mathcal{O}_C$. This will give us p-torsion in cohomology that is controlled by G. In particular, if G itself degenerates in an interesting way from the generic fiber to the special fiber, then the torsion in cohomology will too.

We choose our finite flat group scheme G as follows. Let $E/\mathcal{O}_C$ be a supersingular elliptic curve, meaning that although E is smooth, all of the p(-power)-torsion of E_k is concentrated at one point. Choose a subgroup of $E_C[p^2] \cong (\mathbb{Z}/p^2\mathbb{Z})^2$ that is cyclic of order p^2, and let G be its flat closure across the special fiber. Then G_C is isomorphic to the constant group scheme $\mathbb{Z}/p^2\mathbb{Z}$, but G_k is concentrated in one point, and in fact it is equal to $E[p]$, which is killed by p.

Let $\mathfrak{X}$ be a smooth projective surface over $\mathcal{O}_C$ arising as the quotient of a smooth complete intersection surface by a free action of G. (One has to be a little careful constructing such a complete intersection and an action, but it exists.) From the group cohomology argument outlined earlier, one can show that $H^2_{\text{ét}}(X, \mathbb{Z}_p) \cong \mathbb{Z}/p^2\mathbb{Z}$. But $H^2_{\text{crys}}(\mathfrak{X}_k/W(k))$ turns out to be $k \oplus k$, which is not surprising because G_k "looks like" an infinitesimal copy of $(\mathbb{Z}/p\mathbb{Z})^2$.