The étale specialization of $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$

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1 Introduction and outline

In this talk we will construct the cohomology theory $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$ and identify its étale specialization. This cohomology theory applies to a certain class of adic spaces $\mathfrak{X}$ which are not perfectoid, but it is best understood by studying perfectoid covers. These covers will not be étale, but will be pro-étale (once we define pro-étale). So our strategy will be as follows:

1. Construct the pro-étale site $X_{\text{proét}}$.
2. Construct some interesting sheaves on $X_{\text{proét}}$.
3. Push one of our pro-étale sheaves down to a derived sheaf on the Zariski site.
4. Modify it with $L\eta_\mu$.
5. Take $R\Gamma$.
6. Study the result using input from perfectoid geometry (e.g. the almost purity theorem).

2 The pro-étale site

Let X be a locally Noetherian adic space, i.e. an adic space covered by affinoid opens $\text{Spa}(R, R^+)$ where R is strongly Noetherian or has a Noetherian ring of definition. We want to define the pro-étale site of X.

First, we define an étale morphism of adic spaces by direct analogy to the case of schemes:

*Notes for a talk given in Berkeley's number theory seminar, organized by Kęstutis Česnavičius, Xinyi Yuan, Sug Woo Shin, and Ken Ribet. Main references: *p-adic Hodge theory for rigid-analytic varieties* (and corrigendum), by Peter Scholze; *Integral p-adic Hodge theory*, by Bhatt-Morrow-Scholze; and *Notes on the $\mathbb{A}_{\text{inf}}$-cohomology of Integral p-adic Hodge theory*, by Matthew Morrow.

$X \to Y$ is étale if for all affinoid rings (R, R^+) and square-zero ideals $I \subset R$, the following diagram induces a unique map as indicated:

$$\begin{array}{ccc} \mathrm{Spa}((R, R^+)/I) & \longrightarrow & X \\ \big\downarrow & \overset{\exists!}{\nearrow} & \big\downarrow \\ \mathrm{Spa}(R, R^+) & \longrightarrow & Y \end{array}$$

where $(R, R^+)/I$ denotes $(R/I, \text{integral closure of } R^+/(I \cap R^+) \text{ in } R/I)$. This gives us $X_{\text{ét}}$ as a category: it is the category of all étale morphisms $Y \to X$. We make this into a site by declaring a cover to be a jointly surjective étale maps.

Defining $X_{\text{proét}}$ will take a few tries. A first attempt: given a category $\mathcal{C}$, the category pro-$\mathcal{C}$ is defined to be the full subcategory of $\mathrm{Funct}(\mathcal{C}^{\text{op}}, \mathrm{Set})$ consisting of functors representable by a small cofiltered inverse limit of representable objects.[1] We write objects of pro-$\mathcal{C}$ as $U = \lim_{\leftarrow} U_i$, although we emphasize that we are not asserting that the inverse limit is representable, and that an object in pro-$\mathcal{C}$ does not include the data of such an inverse limit. (We could however do so at the cost of modifying the definition of morphisms.) Notice that we can associate a natural topological space to $U = \lim_{\leftarrow} U_i$, namely $|U| = \lim_{\leftarrow} |U_i|$.

Unfortunately, pro-$X_{\text{ét}}$ is too large: we want pro-étale neighborhoods to look like one étale neighborhood $U_0 \to X$ followed by a cofiltered inverse limit of finite étale surjective maps $U_j \to U_i$, but pro-$X_{\text{ét}}$ allows inverse systems of infinitely many non-finite and non-surjective maps. So instead, we say that $U \in$ pro-$X_{\text{ét}}$ is *pro-étale* over X if U is isomorphic in pro-$X_{\text{ét}}$ to an object of this form, and we let $X_{\text{proét}}$ be the full subcategory of pro-étale objects in pro-$X_{\text{ét}}$.

Finally, we must define the coverings in $X_{\text{proét}}$. This is somewhat hairy due to the possibility of uncountable inverse limits; in fact, Scholze's original definition was wrong and had to be corrected later. (In particular, don't worry too much about the details—we care almost exclusively about the countable case.) We say that a collection of maps $\{f_i : U_i \to U\}$ in $X_{\text{proét}}$ is a covering if it satisfies two conditions. First, the induced maps $|U_i| \to |U|$ must be jointly surjective. Second, for each i, we require that $U_i \to U$ can be written as an inverse limit $\lim_{\leftarrow \mu < \lambda}$ in $X_{\text{proét}}$ along some ordinal λ, with $U_0 \to U$ étale, and such that for all $\mu > 0$, the map $U_\mu \to \lim_{\mu' < \mu} U_{\mu'}$ is finite étale and surjective. (Here, we say that a morphism in $X_{\text{proét}}$ is *étale*, respectively *finite étale and surjective*, if it is the pullback in $X_{\text{proét}}$ of a morphism in $X_{\text{ét}}$ with the same properties.)

Example 1. *Suppose X is defined over $\mathrm{Spa}(K, \mathcal{O}_K)$ with K perfectoid of characteristic 0, and suppose $U \in X_{\text{proét}}$ can be presented as $\lim_{\leftarrow} U_i \to X$, where $U_i = \mathrm{Spa}(R_i, R_i^+)$, $U_0 \to X$ is étale, and all maps $U_i \to U_j$ are finite étale surjective. We say that U is* affinoid perfectoid *if for $R^+ = \widehat{\lim_{\to} R_i^+}$ and $R = R^+[1/p]$, (R, R^+) is a perfectoid affinoid K-algebra. In this case, one can show that $\mathrm{Spa}(R, R^+) \sim \lim_{\leftarrow} \mathrm{Spa}(R_i, R_i^+)$.*

The pro-étale site comes equipped with a morphism of sites $\nu : X_{\text{proét}} \to X_{\text{ét}}$, where $\nu_* :$

[1]Scholze's Definition 3.1 puts the "op" in the wrong place.

$\mathrm{Sh}(X_{\text{proét}}) \to \mathrm{Sh}(X_{\text{ét}})$ forgets all inverse limits that aren't representable by a single étale cover, and ν^* is its left adjoint. As with all morphisms of sites, ν_* is left-exact, and ν^* is exact.[2]

3 Some pro-étale sheaves

Let X now be a locally Noetherian adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. (We will later be interested in X over $\mathrm{Spa}(K, \mathcal{O}_K)$ for $K/\mathbb{Q}_p$ a perfectoid field. The main thing for now is that p should be a topologically nilpotent unit on X.)

We will now define a series of important sheaves on the pro-étale site of X. First, we have structure sheaves

$$\mathcal{O}_X = \nu^*\mathcal{O}_{X_{\text{ét}}} \text{ and} \tag{1}$$

$$\mathcal{O}_X^+ = \nu^*\mathcal{O}_{X_{\text{ét}}}^+, \tag{2}$$

where $\mathcal{O}_{X_{\text{ét}}}(U) = \mathcal{O}_U(U)$, and similarly for $\mathcal{O}_{X_{\text{ét}}}^+$. But these are the wrong things to consider. For example, for $U = \lim_{\leftarrow} \mathrm{Spa}(R_i, R_i^+) \to X$ an affinoid perfectoid neighborhood, then $\mathcal{O}_X(U) = \lim_{\to} R_i$, which is generally not p-adically complete. We want everything to be p-adically complete, so naturally, we take p-adic completions:

$$\widehat{\mathcal{O}}_X^+ = \lim_{\leftarrow r} \mathcal{O}_X^+/p^r, \text{ and} \tag{3}$$

$$\widehat{\mathcal{O}}_X = \widehat{\mathcal{O}}_X^+[1/p]. \tag{4}$$

We similarly have a tilted structure sheaf,

$$\mathcal{O}_X^{+\flat} = \widehat{\mathcal{O}}_X^{+\flat} = \lim_{\leftarrow \varphi} \mathcal{O}_X^+/p; \tag{5}$$

this one doesn't need to be completed, since p is zero on it. Note that all of these constructions are defined on the level of sheaves, and the question of whether their sections on a general $U \in X_{\text{proét}}$ behave as one might guess is subtle.

Finally, since we are trying to construct a cohomology theory over A_{inf}, we define

$$\mathbb{A}_{\text{inf},X} = W(\widehat{O}_X^{+\flat}), \tag{6}$$

by setting $\mathbb{A}_{\text{inf},X}(U) = W(\widehat{O}_X^{+\flat}(U))$; this is already a sheaf. Notice that when we are over $\mathrm{Spa}(K, \mathcal{O}_K)$, $\mathbb{A}_{\text{inf},X}$ is a sheaf of algebras over $W(\mathcal{O}_K^\flat) = A_{\text{inf}}(\mathcal{O}_K)$; this will be a crucial ingredient in the construction of $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$.

Aside: one can also define period sheaves corresponding to B_{inf}, B_{dR}^+, B_{dR}, etc. We won't need these today.

A useful result, analogous to the situation for affine schemes, is the following:

[2]Recall that when defining kernels of sheaf morphisms, we literally just take the kernel on each open set; but when defining cokernels, we need to sheafify. Because of this, "forgetting" from $X_{\text{proét}}$ to $X_{\text{ét}}$ preserves kernels, but it may not preserve cokernels, as the two sheafifications may disagree.

Lemma 2. *If $U \in X_{pro\acute{e}t}$ is affinoid perfectoid, then $H^i(U, \widehat{\mathcal{O}}_X^+)$ is an almost zero $\mathcal{O}_K$-module for $i > 0$.*

4 $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$ and its étale specialization

We are now ready to construct the cohomology theory we've been building up to all semester. As we will see in the coming weeks and months, this has the remarkable property of specializing to all known integral p-adic cohomology theories (of a smooth proper variety over $\mathcal{O}_K$) along various specializations of A_{inf}: p-adic étale cohomology of the generic fiber, crystalline cohomology of the special fiber, and de Rham cohomology of the full integral model over $\mathcal{O}_K$. This will allow us to obtain some interesting comparisons between the torsion in these cohomology theories.

Fix a perfectoid field K of characteristic 0 endowed with a fixed choice of ζ_{p^r} for all r. (The choice is just for convenience; the constructions will end up being independent of it.) Let $\mathcal{O}$ be the ring of integers, with the p-adic topology. Let $\mathfrak{X}/\operatorname{Spf}\mathcal{O} := \operatorname{Spa}(\mathcal{O}, \mathcal{O})$ be a smooth proper formal scheme. (For our purposes, a formal scheme is just an adic space covered by affinoids of the form $\operatorname{Spa}(R, R)$; this is smooth over $\mathcal{O}$ if and only if R is flat and p-adically complete and R/p is smooth over $\mathcal{O}/p$.)

Since $\operatorname{Spa}(\mathcal{O}, \mathcal{O})$ has two points, the generic point $\operatorname{Spa}(K, \mathcal{O})$ (where p is a topologically nilpotent unit) and the special point $\operatorname{Spa}(k, k)$ (where $p = 0$; k has the discrete topology), we can consider the generic fiber X and special fiber $\mathfrak{X}_k$ of $\mathfrak{X}$. Note that the generic fiber X, not the full integral model $\mathfrak{X}$, plays the role of X from the previous sections: in order to define our period sheaves, we had to be on the generic fiber.

We want to turn our pro-étale sheaves on X into Zariski sheaves on $\mathfrak{X}$, so we use the projection $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{Zar}}$ that intersects Zariski open subsets with the generic fiber and views them as pro-étale neighborhoods. (Note that here we're using the Zariski site where earlier we used the étale site. Bhatt-Morrow-Scholze notes that everything would still work if we continued to use the étale site, but Zariski is enough.) Since ν_* is left-exact but not necessarily right-exact, we will also use its derived functor $R\nu_*$ when working on the derived category.

We are now ready to define the object $A\Omega_{\mathfrak{X}}$:

$$A\Omega_{\mathfrak{X}} = L\eta_\mu(R\nu_*\mathbb{A}_{\text{inf},X}) \in D(\mathfrak{X}_{\text{Zar}}). \tag{7}$$

Namely, we push $\mathbb{A}_{\text{inf},X}$ forward from $X_{\text{proét}}$ to $\mathfrak{X}_{\text{Zar}}$, in the derived sense, and then kill off some cohomology by applying $L\eta_\mu$. (Recall the construction of μ: the system ζ_{p^r} gives rise to an element $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^\flat$. Taking the Teichmüller lift and subtracting 1, we get $\mu = [\epsilon] - 1 \in A_{\text{inf}}$. Notice that changing our choice of ζ_{p^r} only changes μ by a unit, so $L\eta_\mu$ is independent of it.) Finally, we define the cohomology theory $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$ by

$$R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) = R\Gamma(\mathfrak{X}_{\text{Zar}}, A\Omega_{\mathfrak{X}}), \tag{8}$$

the derived global sections of $A\Omega_{\mathfrak{X}}$ on the Zariski site.

Informally, $A\Omega_{\mathfrak{X}}$ should be thought of as a "universal deformation" of the cotangent complex $\Omega^\bullet = (\mathcal{O}_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/\mathcal{O}_K} \to \Omega^2_{\mathfrak{X}/\mathcal{O}_K} \to \cdots)$ that lives over A_{inf} instead of $\mathcal{O}_K$. This makes it plausible that specializing $A\Omega_{\mathfrak{X}}$ along $\theta : A_{\mathrm{inf}} \to \mathcal{O}_K$ should recover the de Rham cohomology $R\Gamma(\Omega^\bullet)$. We will make this more precise over the next two talks.

Theorem 3. *$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a perfect complex in $D(A_{\mathrm{inf}})$, and there is a canonical quasi-isomorphism $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] \simeq R\Gamma_{\acute{e}t}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]$. (Recall that the localization $A_{\mathrm{inf}} \to A_{\mathrm{inf}}$ is called the étale specialization of A_{inf}.)*

Proof. (Sketch, assuming $K = \overline{K}$ for the last step.) We omit the proof that the complex is perfect, as this relies on some of the other comparison theorems. The étale comparison comes from a chain of isomorphisms in $D(A_{\mathrm{inf}}[1/\mu])$:

$$\begin{aligned} R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] &\simeq R\Gamma(R\nu_* \mathbb{A}_{\mathrm{inf},X}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] && (9)\\ &\simeq R\Gamma(X_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},X}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] && (10)\\ &\simeq R\Gamma_{\mathrm{\acute{e}t}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]. && (11)\end{aligned}$$

The last step comes from an almost quasi-isomorphism

$$R\Gamma_{\mathrm{\acute{e}t}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \to R\Gamma(X_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf},X}), \tag{12}$$

using almost mathematics with respect to the ideal $W(\mathfrak{m}^\flat)$ of A_{inf}; this becomes an actual quasi-isomorphism once we invert the element $\mu \in W(\mathfrak{m}^\flat)$. □

5 Almost purity theorem

We now sketch how perfectoid covers and the almost purity theorem are useful in the computation of $A\Omega_{\mathfrak{X}}$. First, we state Scholze's version of Faltings' almost purity theorem:

Theorem 4. *If R is a perfectoid K-algebra and S/R is finite étale, then S is a perfectoid K-algebra, and S° is almost finite étale over R°.*

To apply this to the setup of the previous section, we note that locally on $\mathfrak{X}$, there exists an étale[3] map $\mathfrak{X} = \operatorname{Spf} R \to \widehat{\mathbb{G}}_m^d = \operatorname{Spf} \mathcal{O}\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle$. Then we base change along the perfection of the formal torus:

$$R_\infty := R \widehat{\otimes}_{\mathcal{O}\langle T_i^{\pm 1}\rangle} \mathcal{O}\langle T_i^{\pm 1/p^\infty}\rangle. \tag{13}$$

Note that this comes with an action of the Galois group $\mathbb{Z}_p^d$, where we use our choice of ζ_{p^∞}. It follows formally from the almost purity theorem that we have an almost quasi-isomorphism of continuous group cohomology

$$R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p^d, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \to R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{A}_{\mathrm{inf},X}). \tag{14}$$

It turns out that after applying $L\eta_\mu$ to both sides, this miraculously becomes an actual quasi-isomorphism, the right-hand side becomes $R\Gamma_{A_{\mathrm{inf}}}(\operatorname{Spf} R)$, and the left side becomes something that can be computed explicitly.

[3] Is this finite étale?