Universal δ-functors

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Theorem. *Suppose $T = (T^i, \delta^i)$ is a covariant δ-functor from $\mathscr{A}$ to $\mathscr{B}$, and suppose that for all $M \in \mathscr{A}$ and $i > 0$, there exists a monomorphism $M \to A$ such that $T^i(A) = 0$. Then T is universal, in the sense that given any other δ-functor R, any given natural transformation $T^0 \to R^0$ extends uniquely to a morphism of δ-functors $T \to R$.*

Proof. Fix $T = (T^i, \delta^i)$ as in the theorem, let $R = (R^i, \delta'^i)$ be an arbitrary δ-functor, and fix a natural transformation $\eta^0 : T^0 \to R^0$. We claim that there are unique natural transformations $\eta^i : T^i \to R^i$ for all $i > 0$ extending η^0 into a morphism of δ-functors. We will construct the $\eta^i : T^i \to R^i$ by induction on i, with our inductive hypothesis being that there exists a unique choice of natural transformations $\eta^0, \eta^1, \ldots, \eta^i$ agreeing with each other via all possible δ-maps. This is clearly true for $i = 0$.

Now suppose we have a natural transformation $\eta^{i-1} : T^{i-1} \to R^{i-1}$, and every $M \in \mathscr{A}$ injects into some A with $T^i(A) = 0$. We claim that there exists a unique natural transformation $\eta^i : T^i \to R^i$ that agrees with η^{i-1} via δ. Uniqueness isn't so hard: suppose M is any object of $\mathscr{A}$, and apply T and R to some short exact sequence $0 \to M \to A \to A/M \to 0$ with $T^i(A) = 0$. This gives a pair of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & T^{i-1}(A) \arrow[r] \arrow[d, "\eta^{i-1}(A)"] & T^{i-1}(A/M) \arrow[r, "\delta"] \arrow[d, "\eta^{i-1}(A/M)"] & T^i(M) \arrow[r] & T^i(A) = 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & R^{i-1}(A) \arrow[r] & R^{i-1}(A/M) \arrow[r, "\delta'"'] & R^i(M) \arrow[r] & R^i(A) \arrow[r] & \cdots
\end{tikzcd}$$

Then we need the morphism $T^{i-1}(A/M) \to T^i(M) \to R^i(M)$ to equal $T^{i-1}(A/M) \to R^{i-1}(A/M) \to R^i(M)$ to make the middle square commute. This morphism is zero on the image of $T^{i-1}(A)$ by an easy diagram chase, so it defines a unique morphism from $T^i(M) = \mathrm{coker}(T^{i-1}(A) \to T^{i-1}(A/M))$ to $R^i(M)$. We take this as our definition of the morphism $\eta^i(M) : T^i(M) \to R^i(M)$. Before we proceed, we must show that this is well-defined; that is, it does not depend on the choice of monomorphism $M \to A$. To do this, first suppose we have a morphism of short exact sequences of the form

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & B/M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & A \arrow[r] & A/M \arrow[r] & 0,
\end{tikzcd}$$

where $T^i(A) = T^i(B) = 0$. (We will soon choose B to be $A \times A'$, where A and A' are arbitrary.) We apply T and R to this morphism of short exact sequences to get the first of several three-dimensional diagrams that we'll use in this proof:

$$\begin{tikzcd}
T^{i-1}(B/M) \arrow[rr, "\delta"] \arrow[dr] \arrow[dd] & & T^i(M) \arrow[rr] \arrow[dr, equal] \arrow[dd] & & T^i(B) = 0 \\
& T^{i-1}(A/M) \arrow[rr, "\delta"] \arrow[dd] & & T^i(M) \arrow[dd] & \\
R^{i-1}(B/M) \arrow[rr, "\delta'"] \arrow[dr] & & R^i(M) \arrow[dr, equal] & & \\
& R^{i-1}(A/M) \arrow[rr, "\delta'"] & & R^i(M) &
\end{tikzcd}$$

We want the two morphisms $T^i(M) \to R^i(M)$ to be equal, which just means that the right face of this cube commutes. The top and bottom faces commute, by naturality of δ and δ'. The front and back faces commute, by construction of the two maps $\eta^i(M) : T^i(M) \to R^i(M)$. The left face commutes by naturality of η^{i-1}. So five of the six faces commute, which implies that all six paths from the top back left corner to the bottom front right corner are equal. Since $T^{i-1}(B/M) \to T^i(M)$ is an epimorphism, it follows that the right face commutes.

Now let $M \to A$ and $M \to A'$ be arbitrary monomorphisms, subject to the condition that $T^i(A) = T^i(A') = 0$. These induce a natural map $M \to A \times A'$, which is a monomorphism because the monomorphism $M \to A$ factors as $M \to A \times A' \xrightarrow{\mathrm{pr}_1} A$. Since finite direct products equal finite direct sums in abelian categories, we have a natural short exact sequence $0 \to A \to A \times A' \to A' \to 0$. This gives a long exact sequence $\cdots \to T^i(A) \to T^i(A \times A') \to T^i(A') \to \cdots$, which implies that $T^i(A \times A') = 0$. So we can indeed choose $B = A \times A'$ above, using the morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & A \times A' \arrow[r] \arrow[d, "\mathrm{pr}_1"] & (A \times A')/M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & A \arrow[r] & A/M \arrow[r] & 0
\end{tikzcd}$$

This proves that defining η^i using $M \to A$ is equivalent to doing so using $M \to A \times A'$, which in turn is equivalent to using $M \to A'$. So we have proved well-definition.

Next, we will prove that η^i agrees with η^{i-1} via all possible δ maps. (This is true by definition for δ maps that come from short exact sequences of the form $0 \to M \to A \to A/M \to 0$ as above, but we don't yet know it in general.) Take any short exact sequence $0 \to M \to N \to N/M \to 0$, and choose an injection $N \to A$ with $T^i(A) = 0$. Then we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & N/M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & A \arrow[r] & A/M \arrow[r] & 0
\end{tikzcd}$$

Applying T and R to this short exact sequence, we get a three-dimensional diagram which reads in part:

$$\begin{tikzcd}
T^{i-1}(N/M) \arrow[rr, "\delta"] \arrow[dr] \arrow[dd] & & T^i(M) \arrow[dr, equal] \arrow[dd] & \\
& T^{i-1}(A/M) \arrow[rr, "\delta"] \arrow[dd] & & T^i(M) \arrow[dd] \\
R^{i-1}(N/M) \arrow[rr, "\delta'"] \arrow[dr] & & R^i(M) \arrow[dr, equal] & \\
& R^{i-1}(A/M) \arrow[rr, "\delta'"] & & R^i(M)
\end{tikzcd}$$

(The vertical arrows come from η^{i-1} and η^i.) We want the back square of the cube to commute. The top and bottom faces commute because (T, δ) and (R, δ') are δ-functors. The right face commutes because both maps are $\eta^1(M)$, which we just showed is well-defined. The left face commutes by naturality of η^{i-1}. The front face commutes by the construction of $\eta^i(M)$. So all six possible paths from the top back left corner to the bottom front right corner are equal. It follows that the back face commutes, because $R^i(M) \to R^i(M)$ is the identity map. So we have shown that η^i agrees with η^{i-1} via all possible δ maps.

Finally, we must show that η^i is a natural transformation. As before, let $0 \to M \to N \to N/M \to 0$ be any short exact sequence, and let $N \to A$ be a monomorphism with $T^i(A) = 0$. This gives us three more short exact sequences: $0 \to N \to A \to A/N \to 0$, $0 \to M \to A \to A/M \to 0$, and $0 \to N/M \to A/M \to A/N \to 0$. In the following diagram, we use portions of the long exact sequences of T and R coming from all four of these short exact sequences:

$$\begin{tikzcd}
T^{i-1}(A/M) \arrow[rr] \arrow[dr] \arrow[dd] & & T^{i-1}(A/N) \arrow[rr] \arrow[dr] \arrow[dd] & & T^i(N/M) \arrow[dr] \arrow[dd, equal] & \\
& R^{i-1}(A/M) \arrow[rr] \arrow[dd] & & R^{i-1}(A/N) \arrow[rr] \arrow[dd] & & R^i(N/M) \arrow[dd, equal] \\
T^i(M) \arrow[rr] \arrow[dr] \arrow[dd] & & T^i(N) \arrow[rr] \arrow[dr] \arrow[dd] & & T^i(N/M) \arrow[dr] & \\
& R^i(M) \arrow[rr] & & R^i(N) \arrow[rr] & & R^i(N/M) \\
0 = T^i(A) \arrow[rr, equal] & & T^i(A) = 0 & & &
\end{tikzcd}$$

(The "back to front" arrows come from η^{i-1} and η^i. Within the T- and R-planes, the left square comes from mapping the short exact sequence $0 \to M \to A \to A/M \to 0$ to $0 \to N \to A \to A/N \to 0$, and applying T and R. The left-to-right rows come from applying T and R to the short exact sequences $0 \to N/M \to A/M \to A/N \to 0$ and $0 \to M \to N \to N/M \to 0$.) Our goal is to show that the bottom faces of the two cubes are commutative. If this is the case, then the "back to front" morphisms η^i are natural for the (arbitrary) monomorphism $M \to N$, and also for the (arbitrary) epimorphism $N \to N/M$. Since every morphism can be factored as a composition of a monomorphism with an epimorphism, it will follow that η^i is natural for all morphisms.

Notice that since $T^{i-1}(A/M) \to T^i(M)$ and $T^{i-1}(A/N) \to T^i(N)$ are epimorphisms, it suffices by the usual three-dimensional diagram-chasing argument to prove that all faces of the two cubes other than the bottom ones commute. Most of these are trivial, but I'll briefly justify them. The far right square commutes because both morphisms are $\eta^i(N/M)$; the squares parallel to it commute because η^i agrees with η^{i-1} via δ. The top plane commutes by the same fact, and by the naturality of η^{i-1}. The front and back faces of the left cube commute by naturality of δ and δ'.

This leaves the front and back faces of the right cube. We will show that the front face commutes; the back face follows by the same argument. Consider the morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A/N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N/M \arrow[r] & A/M \arrow[r] & A/N \arrow[r] & 0.
\end{tikzcd}$$

Applying R to this diagram, we get a ladder of morphisms between long exact sequences, where everything commutes by the naturality of δ'. One square in this diagram is as follows:

$$\begin{tikzcd}
R^{i-1}(A/N) \arrow[r, "\delta'"] \arrow[d, equal] & R^i(N) \arrow[d] \\
R^{i-1}(A/N) \arrow[r, "\delta'"] & R^i(N/M)
\end{tikzcd}$$

If we contract the equals signs and squint our eyes a little, we see that this is the same as the square in question. So we have proved commutativity of all necessary squares in the big diagram. It follows that the bottom faces in that diagram commute, so η^i is natural for both monomorphisms and epimorphisms, and thus for all morphisms.

We have now finished the inductive step of our proof: given a natural transformation $\eta^{i-1} : T^{i-1} \to R^{i-1}$, we have constructed a natural transformation $\eta^i : T^i \to R^i$ that agrees with η^{i-1} via δ, and we have shown that it is unique. So putting all the η^i together, we get a unique morphism η of δ-functors $T \to R$ extending the given transformation $\eta^0 : T^0 \to R^0$, as desired. So (T, δ) is indeed a universal δ-functor.

$\square$