Split fibered categories

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Recently, we've been talking a lot about fibered categories, and we've repeatedly mentioned a peculiar[1] fact: that although not every fibered category admits a splitting, every fibered category $F \to C$ *is* equivalent (over C) to a split fibered category $\widetilde{F} \to C$. To see how such an equivalence of categories can work out explicitly, let's look at a very concrete example of a fibered category: an extension of groups.

1 The case of groups

Let $f : G \to H$ be a surjective group homomorphism, and consider the corresponding functor $\phi : BG \to BH$, where these categories consist of one object whose endomorphisms are identified with the groups G and H, respectively. Since every morphism in these categories is an isomorphism, every commutative square is cartesian, so we have a fibered category by surjectivity of f. Notice, though, that there exists a splitting if and only if the surjection $G \to H$ splits—this follows immediately from unrolling the definition of a splitting of a fibered category. Since it is easy to write down a non-split group extension, it is natural to ask how to modify such a thing to turn it into a split fibered category.

The following construction was inspired by our discussion of the case $\mathbb{Z}/4\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ in class, but it works for any extension of groups. Let $\widetilde{F}$ be the category whose objects are indexed by H, whose hom-sets are $\mathrm{Hom}_{\widetilde{F}}(h_1, h_2) = G$ for all h_1, h_2, and whose composition is given by

$$h_1 \xrightarrow{g_1} h_2 \xrightarrow{g_2} h_3, \qquad h_1 \xrightarrow{g_1 g_2} h_3$$

(It is admittedly strange to have both objects and morphisms indexed by group elements, and to give the same names to different morphisms between different objects, but hopefully the meaning is clear.) We claim that $\widetilde{F}$ can be given the structure of a split fibered category over BH that is equivalent to BG as a fibered category.

[1] Personally, I find this weird because it feels like a counterexample to an intuition that I've had ever since I started learning abstract algebra: that isomorphic things differ only in name. The explanation, of course, is that an equivalence of categories is only a weak form of "isomorphism": composing an equivalence of categories with its (quasi-)inverse yields a functor naturally equivalent, but not necessary equal to, the identity functor.

To prove our claim, we must first define some functors. Let $\pi : \widetilde{F} \to BG$ be the "projection" functor that sends all objects to the unique object $*$ of BG, and sends every arrow $h : g_1 \to g_2$ to the arrow of the same name in BG. Then the composition $\phi \circ \pi$ expresses $\widetilde{F}$ as a category over BH. In fact, $\widetilde{F}$ is clearly a fibered category: any morphism $h : * \to *$ in BH pulls back to $\overline{h} : h_1 \to h_2$ for all objects $h_1, h_2 \in \widetilde{F}$ and all lifts $\overline{h} \in G$ of h.

We claim that $\widetilde{F}$ is a split fibered category, and that it is equivalent to BG as a fibered category over BH. The equivalence is easy: in one direction, we have the functor $\pi : \widetilde{F} \to BG$, and we can include BG back into $\widetilde{F}$ by mapping the object $* \in BG$ to (say) $e \in \widetilde{F}$ and preserving the labels of morphisms. The composition $BG \to \widetilde{F} \to BG$ is the identity functor, and the composition $\widetilde{F} \to BG \to \widetilde{F}$ is the functor of projection onto the object e. This is naturally equivalent to the identity functor via the base-preserving natural transformation $h \xrightarrow{e} e$.

To construct a splitting, fix a representative $\widetilde{h} \in G$ for each element $h \in H$. Consider the subcategory of $\widetilde{F}$ that includes all objects, but only the morphisms of the form

$$h_1 \xrightarrow{\widetilde{h}_1 \widetilde{h}_2^{-1}} h_2.$$

First of all, this is a subcategory: it clearly contains all identity maps, and it contains compositions according to the commutative diagram

$$h_1 \xrightarrow{\widetilde{h}_1 \widetilde{h}_2^{-1}} h_2 \xrightarrow{\widetilde{h}_2 \widetilde{h}_3^{-1}} h_3, \qquad h_1 \xrightarrow{\widetilde{h}_1 \widetilde{h}_3^{-1}} h_3$$

But for a given morphism h in BH and any object $h_2 \in \widetilde{F}$, our chosen subcategory contains a unique pullback $h_1 \to h_2$ of h, namely the morphism $\widetilde{h}_1 \widetilde{h}_2^{-1} : h_1 \to h_2$ with $h_1 = hh_2$ chosen so that $h_1 h_2^{-1} = h$.

2 The general construction

Before constructing splittings, we recall some preliminary constructions. Let C be any category. For any $U \in C$, let C/U denote the comma category, whose objects are morphisms to U and whose morphisms are commutative triangles. This admits a natural functor to C, given by sending $X \to U$ to X, and similarly for morphisms. In fact, this functor $C/U \to C$ expresses C/U as a fibered category: given any object $W \to U$ in C/U and a morphism $V \to W$ in C, we can lift $V \to W$ to the obvious morphism in C/U:

$$\begin{array}{ccc} V & \longrightarrow & W \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Next, given a morphism $f : U \to V$ in C, we get a functor $f \circ - : C/U \to C/V$ sending $X \to U$ to the composition $X \to U \to V$, and similarly for morphisms. To prove this is a morphism of fibered categories, observe first that it clearly commutes with the projection functors to C. We also need to show that it sends cartesian arrows to cartesian arrows; I'll fill this part in later.

Now let $p : F \to C$ be any fibered category. We construct $\widetilde{F}$ as follows. The objects of $\widetilde{F}$ are morphisms of fibered categories $C/U \to F$, where U ranges over the objects of C. The morphisms from $\phi : C/U \to F$ to $\phi' : C/V \to F$ are commutative diagrams of morphisms of fibered categories

$$\begin{array}{ccc} C/U & \longrightarrow & C/V \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & F & \end{array}$$

over C.

We claim that this category is fibered over C, that it admits a natural splitting, and that it is equivalent to F as a fibered category. Let's first construct a functor $\widetilde{F} \to F$. Send the object $\phi : C/U \to F$ in $\widetilde{F}$ to $\phi(\mathrm{id}_U) \in F$. Given a morphism $C/U \to C/V$ in $\widetilde{F}$, recall from weak 2-Yoneda that this can be written as $f \circ -$ for a unique $f : U \to V$. We want a morphism from $\phi(\mathrm{id}_U)$ to $\phi(\mathrm{id}_V)$ in F.

To construct the splitting, fix a morphism $f : U \to V$ in C. Then we get a natural functor $\mathrm{Hom}_C(C/V, F) \to \mathrm{Hom}_C(C/U, F)$ given by precomposing with $f \circ - : C/U \to C/V$. If $\phi' \in \mathrm{Hom}_C(C/V, F)$ is given, then the resulting $\phi : C/U \to C/V \to F$ comes with the obvious morphism to ϕ' in $\widetilde{F}$:

$$\begin{array}{ccc} C/U & \xrightarrow{f\circ -} & C/V \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & F & \end{array}$$