Special values of L-functions

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Riemann zeta function

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$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $n^s := e^{s \log n}$. This series is absolutely convergent, so it gives a well-defined function from a half-plane of $\mathbb{C}$ to $\mathbb{C}$.

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$$= \prod_p \frac{1}{1 - 1/p^s}.$$

One must take some care when working with infinite products of infinite sums like this, but everything does converge correctly.

Analytic continuation

Riemann proved that the $\zeta(s)$ has an *analytic continuation* to $\mathbb{C}\setminus\{1\}$; that is, a (necessarily unique) holomorphic function $\zeta : \mathbb{C}\setminus\{1\} \to \mathbb{C}$ that restricts to our previously defined function on the region $\mathrm{Re}(s) > 1$.

Near $s = 1$, $\zeta(s)$ blows up as $\frac{1}{s-1} + \gamma + O(s-1)$, where $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

Functional equation

Riemann also proved a *functional equation*:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

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Here, Γ is the gamma function, given by (the analytic continuation of)

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t.$$

The gamma function is much easier to understand than the zeta function; one of the breakout sessions this afternoon will study its properties.

Special values of $\zeta(s)$

The zeta function encodes arithmetic information about the integers. Let's look at some of its particular values.

n	-4	-3	-2	-1	0	1	2	3	4	5
$\zeta(n)$	0	$\frac{1}{120}$	0	$-\frac{1}{12}$	$-\frac{1}{2}$	pole	$\frac{\pi^2}{6}$	≈ 1.202	$\frac{\pi^4}{90}$	≈ 1.037

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- For $n > 0$ even, $\zeta(n)$ is determined by $\zeta(1-n)$ via the functional equation; it is a nonzero rational multiple of π^{2n}.
- The values at positive odd integers seem to have no simple formula.

Special values of $\zeta(s)$, continued

Rearranging the functional equation gives:

$$\zeta(s) = \frac{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}$$

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(Here we are using the fact that Γ has poles at $0, -1, -2, \ldots$.)

Dirichlet characters

We now take a step back and modify the definition of $\zeta(s)$ to construct a new function $L(\chi, s)$, which will turn out to encode information about a larger field than $\mathbb{Q}$, in this case $\mathbb{Q}(i)$. Although we will focus on one function for concreteness, we emphasize that the following discussion can be made much more general.

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For $n \in \mathbb{Z}$, define

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if n is even.} \end{cases}$$

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This is a *Dirichlet character*; i.e. a completely multiplicative function on $\mathbb{Z}$ that is periodic mod some m (here $m = 4$) and equal to zero on inputs that are not relatively prime to m.

Dirichlet L-function $L(\chi, s)$

We then define our L-function by the Dirichlet series

$$\begin{aligned} L(\chi, s) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\ &= 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots . \end{aligned}$$

This again converges absolutely for $\mathrm{Re}(s) > 1$.

Euler product for $L(\chi, s)$

As with ζ, this has an Euler product, but now different primes behave differently in it:

$$L(\chi, s) = \prod_{p=2} 1 \cdot \prod_{p \equiv 1 \pmod 4} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$
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$L(\chi, s)$ and $\mathbb{Q}(i)$

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L(\chi, s)\zeta(s) &= \prod_{p=2} \frac{1}{1 - 1/p^s} \cdot \prod_{p \equiv 1 \pmod 4} \left(\frac{1}{1 - 1/p^s} \right)^2 \\
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$L(\chi, s)$ and $\mathbb{Q}(i)$, continued

This is the natural analogue of $\zeta(s)$ for the number field $\mathbb{Q}(i)$, so we will call this function $\zeta_{\mathbb{Q}(i)}(s)$. So we have shown the identity:

$$L(\chi, s) = \frac{\zeta_{\mathbb{Q}(i)}(s)}{\zeta(s)}.$$

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Remark: We can analogously define the Dedekind zeta function ζ_K for an arbitrary number field K. When $K/\mathbb{Q}$ is an abelian (Galois) extension, ζ_K will factor as a product of $[K : \mathbb{Q}]$ Dirichlet L-functions, including $\zeta(s)$ itself.

Features of $L(\chi, s)$

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- The generalized Riemann hypothesis (open problem), which predicts that all zeroes of $L(\chi, s)$ with real part between 0 and 1 must have real part $\frac{1}{2}$, for all Dirichlet characters χ.

Special values of $L(\chi, s)$

n	-4	-3	-2	-1	0	1	2	3	4	5
$L(\chi, n)$	$\frac{5}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\pi}{4}$	≈ 0.916	$\frac{\pi^3}{32}$	≈ 0.989	$\frac{5\pi^5}{1536}$

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- The roles of negative evens and negative odds have been switched, and similarly positive evens and positive odds. This is because the gamma-factors in the functional equation have poles in different places.
- The value at 1 is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$, which we will now see has a particularly special meaning.

Analytic class number formula

Theorem (Dedekind 1894, Landau 1903)

For any number field K, we have:

$$\lim_{s \to 1} \frac{\zeta_K(s)}{\zeta(s)} = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|D_K|}},$$

where:

- r_1 is the number of embeddings $K \hookrightarrow \mathbb{R}$,
- r_2 is the number of conjugate pairs of embeddings $K \hookrightarrow \mathbb{C}$ with image not contained in $\mathbb{R}$,
- $h_K = |\mathrm{Cl}_K|$ is the class number of K,
- R_K is the regulator of K,
- w_K is the number of roots of unity in K, and
- D_K is the discriminant of K.

Analytic class number formula, continued

In our situation, $K = \mathbb{Q}(i)$:

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&= \frac{2^0 \cdot (2\pi)^1 \cdot 1 \cdot 1}{4 \cdot \sqrt{|-4|}}
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L-functions in general

There exist much more general things that deserve to be called L-functions. These are functions built from Dirichlet series (or Euler products), with terms of arithmetic or algebro-geometric interest. In general we expect them to have:

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- a *Riemann hypothesis* restricting the location of their zeroes and poles, and
- rational values—or more precisely, values that are rational multiples of some predictable "periods" such as powers of π—at all integer inputs where neither of the Γ-factors in the functional equation has a pole. (Deligne's conjecture on special values)

Example: the L-function of an elliptic curve

If $E = (y^2 = x^3 + ax + b)$ is an elliptic curve over $\mathbb{Q}$, let

$$a_p = p + 1 - \#E(\mathbb{F}_p)$$

for each prime p, where $\#E(\mathbb{F}_p)$ is the number of points of the reduction of (a minimal Weierstrass model of) E modulo p.

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for each prime p, where $\#E(\mathbb{F}_p)$ is the number of points of the reduction of (a minimal Weierstrass model of) E modulo p. The L-function of E is defined approximately as

$$L(E, s) = \prod_p \left(1 - a_p \cdot p^{-s} + p \cdot p^{-2s}\right)^{-1},$$

with suitable corrections at the finitely many primes of bad reduction.

Birch and Swinnerton-Dyer conjecture

Conjecture (Birch and Swinnerton-Dyer)

For every elliptic curve $E/\mathbb{Q}$,

$$\operatorname{ord}_{s=1} L(E,s) = \operatorname{rank} E(\mathbb{Q}).$$

Moreover, there is a formula for the leading coefficient in terms of arithmetic invariants of the curve:

$$\lim_{s \to 1} \frac{L(E,s)}{(s-1)^r} = \frac{|\text{Ш}_{E/\mathbb{Q}}| \cdot \Omega_E \cdot R_E \cdot \prod_{p|2\Delta} c_p}{|E(\mathbb{Q})_{\text{tors}}|^2}.$$