Proof of lemma:

Let $D^\bullet$ be a perfect complex of R-modules. Up to quasi-isomorphism, we write $D^\bullet$ as a bounded complex of finite projective, thus free, modules. Suppose we know

$$\dim_K H^i(D^\bullet \otimes_R K) \text{ and}$$
$$\dim_k H^i(D^\bullet \otimes_R k) \text{ for all } i.$$

We want to understand the torsion in the integral cohomology $H^i(D^\bullet)$. Note that

$$H^i(D^\bullet \otimes_R k) \cong (H^i(D^\bullet) \otimes_R k) \oplus \mathrm{Tor}_1^R(H^{i+1}(D^\bullet), k)$$

non-canonically by the Universal Coefficient Theorem.

We can check that each $H^i(D^\bullet)$ is a finitely presented module, so we have an isom.

$$H^{i+1}(D^\bullet) \cong R^{\oplus a} \oplus \bigoplus_{j=1}^{b} (R/\alpha_j),$$

for some $0 \neq \alpha_j \in \mathfrak{m}$.

Using the resolution

$$0 \to \bigoplus_{j=1}^{b} R \xrightarrow{\alpha_j} \begin{array}{c} R^{\oplus a} \\ \oplus \\ \bigoplus_{j=1}^{b} R \end{array} \to H^{i+1}(D^\bullet) \to 0,$$

we compute

$$\begin{aligned} \mathrm{Tor}_1^R(H^{i+1}(D^\bullet), k) &= \ker\left(k^{\oplus b} \xrightarrow{\alpha_j = 0} k^{\oplus a+b}\right) \\ &= k^{\oplus b}. \end{aligned}$$

This dimension b can be rewritten as:

$$b = \dim_k\left(H^{i+1}(D^\bullet) \otimes_R k\right) \overset{= a+b}{} \\ - \dim_K\left(H^{i+1}(D^\bullet \otimes_R K)\right). \overset{= a}{}$$

So putting everything together, we have:

$$\begin{aligned} \dim_k(H^i(D^\bullet \otimes k)) &= \dim_k(H^i(D^\bullet) \otimes k) \\ &\quad + \dim_k(\mathrm{Tor}_1^R(H^{i+1}(D^\bullet), k)) \\ &= \dim_k(H^i(D^\bullet) \otimes k) \\ &\quad + \dim_k(H^{i+1}(D^\bullet) \otimes k) \\ &\quad - \dim_K(H^{i+1}(D^\bullet \otimes K)). \end{aligned}$$

Since $D^\bullet$ is bounded, this determines $\dim_k(H^i(D^\bullet) \otimes k)$ for all i, and therefore also $\dim_k(H^i(D^\bullet)_{\mathrm{tors}} \otimes k) = \dim_k(H^i(D^\bullet) \otimes k) - \dim_K(H^i(D^\bullet \otimes K))$.