Extra algebraic points in $X(\mathbb{Q}_p)_1$

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June 18, 2021

Background

In Lecture 2, we saw some examples of genus-3 hyperelliptic curves $X/\mathbb{Q}$ with $r = \mathrm{rk}\,\mathrm{Jac}\,X = 1$ such that the Chabauty-Coleman set $X(\mathbb{Q}_p)_1$ (for some prime p) does not equal $X(\mathbb{Q})$, but all of the extra points are algebraic and can be explained geometrically.

The simplest way this can happen is when we have a non-rational Weierstrass point $P = (x, 0)$. Then $2(P - \infty)$ is a principal divisor, which forces $\int_\infty^P \omega = 0$ for all global 1-forms ω. More generally, $P \in X(K)$ will lie in $X(\mathbb{Q}_p)_1$ if $n(P - \infty) \in J(\mathbb{Q})$ for some $n > 0$.

Background

A more complicated situation can occur if $J = \text{Jac}(X)$ decomposes as a product of an elliptic curve E and an abelian variety A, all of $J(\mathbb{Q})$ (up to torsion) lies in E, and P lives in E mod torsion. Then we have $P \in \overline{J(\mathbb{Q})} + J(K)_{\text{tors}}$, so P is killed by the annihilating differentials.

Question

Our question was whether genus-2 curves can exhibit similar behavior. That is: can we exhibit a genus-2 rank-1 curve $X/\mathbb{Q}$ and a prime p such that $X(\mathbb{Q}_p)_1$ consists entirely of algebraic points, and such that we can explain the points (other than rational points and Weierstrass points) via a decomposition of $\mathrm{Jac}(X)$?

Answer: yes!

Example

Let X be the genus-2 hyperelliptic curve with LMFDB label 15360.h.184320.1. This is the unique curve on LMFDB with $g = 2$, $r = 1$, $\mathrm{Aut}(X) = V_4$, and four rational Weierstrass points. It is given by the equation

$$y^2 = 2x^5 - x^4 - 5x^3 + 3x + 1,$$

which we renormalized ($4y \mapsto y, 2x \mapsto x$) to the monic form

$$y^2 = x^5 - x^4 - 10x^3 + 24x + 16.$$

Example

This curve has six rational points: $(0, \pm 4)$, $(-2, 0), (-1, 0)$, $(2, 0)$, and ∞. Choosing $p = 7$ (the smallest prime of good reduction),we calculated that $|X(\mathbb{Q}_7)_1| = 14$. Besides the rational points, it contains:

$$P_1 = (\sqrt{2}, 2 + 2\sqrt{2}),$$
$$P_2 = (2 + 2\sqrt{2}, 16 + 12\sqrt{2}),$$

and their orbits under $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and the hyperelliptic involution.

Extra automorphisms

Before explaining the extra points, let's first take a closer look at X. As I said, $\mathrm{Aut}(X)$ is a Klein four-group, where one of the involutions is the hyperelliptic involution ι. Another involution is given by

$$\sigma : (x, y) \mapsto \left(\frac{-2x}{x+2}, \frac{8y}{(x+2)^3} \right).$$

The Jacobian $J = \mathrm{Jac}(X)$ splits (up to isogeny) as $E_1 \times E_2$, where E_1 and E_2 are respectively the quotients of X by the involutions σ and $\sigma \circ \iota$.

(Aside: the quotient by ι is just $\mathbb{P}^1$, of course. Here's a nice visualization challenge: picture a 2-holed torus, and figure out which of the three "obvious" involutions corresponds to the hyperelliptic involution.)

Explanation of extra points

Let's figure out where the (non-torsion) rational points are on $J \sim E_1 \times E_2$.

LMFDB tells us that $J(\mathbb{Q}) \simeq \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$, and $(0, 4)$ and $(0, -4) \in X(\mathbb{Q})$ map to an inverse pair of generators of $J(\mathbb{Q})/\mathrm{tors}$. Notice that σ fixes each of these points, and ι switches them. It follows that the quotient $X/(\sigma \circ \iota) = E_2$ identifies these two points with each other, and in particular $2 \cdot (0, 4)$ maps to 0 in E_2.

So up to torsion, all of $J(\mathbb{Q})$ lives in (and is dense in) the E_1 factor, and none of it lives in E_2.

Explanation of extra points

The upshot: if we can show that P_1 and P_2 map to torsion points in E_2 (i.e. they live in $E_1 + \text{tors} \subset J$), then this will explain their appearance in $X(\mathbb{Q}_7)_1$: every point in E_1 is killed by the annihilating differentials because $J(\mathbb{Q})$ is dense in E_1, and torsion points in J are killed by all differentials.

So we must show that P_1 is torsion in J once we identify every point with its image under $\sigma \circ \iota$, and similarly for P_2. In fact, we have $\sigma \circ \iota(P_1) = \overline{P_2}$ (where the bar denotes the nontrivial element of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$), and similarly $\sigma \circ \iota(P_2) = \overline{P_1}$. A calculation in Sage shows that

$$4(P_1 + \overline{P_2}) = 4(P_2 + \overline{P_1}) = 0 \in J.$$

So P_1 and P_2 are at worst 8-torsion in E_2.

Other primes

Aside: working with the same curve X but $p = 11$ instead, we lose these special points (since $\sqrt{2} \notin \mathbb{Q}_{11}$), gain the irrational Weierstrass points $(1 \pm \sqrt{5}, 0)$, and also gain two pairs of points that appear to be transcendental.

With $p = 17$ (the next prime such that $\sqrt{2} \in \mathbb{Q}_p$), we get the two special points again, along with some points whose x-coordinates appear to lie in the quartic fields 4.2.1984.1 and 4.0.656.1. I don't know why.