# Extra algebraic points in $X(\mathbb{Q}_p)_1$

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# Background

In Lecture 2, we saw some examples of genus-3 hyperelliptic curves $X/\mathbb{Q}$ with $r = \mathrm{rk}\,\mathsf{Jac}\,X = 1$ such that the Chabauty-Coleman set $X(\mathbb{Q}_p)_1$ (for some prime $p$) does not equal $X(\mathbb{Q})$, but all of the extra points are algebraic and can be explained geometrically.

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The simplest way this can happen is when we have a non-rational Weierstrass point $P = (x, 0)$. Then $2(P - \infty)$ is a principal divisor, which forces $\int_\infty^P \omega = 0$ for all global 1-forms $\omega$.

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A more complicated situation can occur if $J = \mathrm{Jac}(X)$ decomposes as a product of an elliptic curve $E$ and an abelian variety $A$, all of $J(\mathbb{Q})$ (up to torsion) lies in $E$, and $P$ lives in $E$ mod torsion. Then we have $P \in \overline{J(\mathbb{Q})} + J(K)_{\mathrm{tors}}$, so $P$ is killed by the annihilating differentials.

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Answer: yes!

# Example

Let $X$ be the genus-2 hyperelliptic curve with LMFDB label 15360.h.184320.1. This is the unique curve on LMFDB with $g = 2$, $r = 1$, $\mathrm{Aut}(X) = V_4$, and four rational Weierstrass points. It is given by the equation

$$y^2 = 2x^5 - x^4 - 5x^3 + 3x + 1,$$

which we renormalized $(4y \mapsto y, 2x \mapsto x)$ to the monic form

$$y^2 = x^5 - x^4 - 10x^3 + 24x + 16.$$

# Example

This curve has six rational points: $(0, \pm 4)$, $(-2, 0), (-1, 0)$, $(2, 0)$, and $\infty$. Choosing $p = 7$ (the smallest prime of good reduction),we calculated that $|X(\mathbb{Q}_7)_1| = 14$. Besides the rational points, it contains:

$$P_1 = (\sqrt{2}, 2 + 2\sqrt{2}),$$
$$P_2 = (2 + 2\sqrt{2}, 16 + 12\sqrt{2}),$$

and their orbits under $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and the hyperelliptic involution.

Before explaining the extra points, let's first take a closer look at $X$. As I said, $\mathrm{Aut}(X)$ is a Klein four-group, where one of the involutions is the hyperelliptic involution $\iota$. Another involution is given by

$$\sigma : (x, y) \mapsto \left( \frac{-2x}{x+2}, \frac{8y}{(x+2)^3} \right).$$

The Jacobian $J = \mathrm{Jac}(X)$ splits (up to isogeny) as $E_1 \times E_2$, where $E_1$ and $E_2$ are respectively the quotients of $X$ by the involutions $\sigma$ and $\sigma \circ \iota$.

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So up to torsion, all of $J(\mathbb{Q})$ lives in (and is dense in) the $E_1$ factor, and none of it lives in $E_2$.

# Explanation of extra points

The upshot: if we can show that $P_1$ and $P_2$ map to torsion points in $E_2$ (i.e. they live in $E_1 + \text{tors} \subset J$), then this will explain their appearance in $X(\mathbb{Q}_7)_1$: every point in $E_1$ is killed by the annihilating differentials because $J(\mathbb{Q})$ is dense in $E_1$, and torsion points in $J$ are killed by all differentials.

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$$4(P_1 + \overline{P_2}) = 4(P_2 + \overline{P_1}) = 0 \in J.$$

So $P_1$ and $P_2$ are at worst 8-torsion in $E_2$.

# Other primes

Aside: working with the same curve $X$ but $p = 11$ instead, we lose these special points (since $\sqrt{2} \notin \mathbb{Q}_{11}$), gain the irrational Weierstrass points $(1 \pm \sqrt{5}, 0)$, and also gain two pairs of points that appear to be transcendental.

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With $p = 17$ (the next prime such that $\sqrt{2} \in \mathbb{Q}_p$), we get the two special points again, along with some points whose $x$-coordinates appear to lie in the quartic fields 4.2.1984.1 and 4.0.656.1. I don't know why.