MATH 53 DISCUSSION SECTION PROBLEMS – 4/27/23

1. The divergence theorem

(1) **(textbook 16.9.5)** Find $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where $\mathbf{F} = xye^z\mathbf{i} + xy^2z^3\mathbf{j} - ye^z\mathbf{k}$ and S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$.

(2) **(textbook 16.9.31)** Suppose S and E satisfy the conditions of the divergence theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint\limits_S f\mathbf{n}dS = \iiint\limits_E \nabla f dV.$$

2. More Stokes' theorem

(3) **(from an old exam, continuation from last Tuesday)** We define the Laplacian of $\mathbf{F}$, the vector field denoted by $\nabla^2\mathbf{F}$, as

$$\nabla^2\mathbf{F} = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R\rangle,$$

where

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is the ordinary Laplacian on scalar-valued functions. It is a fact (which you don't need to prove) that

$$\text{curl}(\text{curl}\,\mathbf{F}) = \text{grad}(\text{div}\,\mathbf{F}) - \nabla^2\mathbf{F}$$

for all vector fields $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ in three dimensions such that all second partial derivatives of P, Q, and R exist. Let $\mathbf{S}$ be a smooth orientable surface with an orientation chosen, let C be its smooth, positively-oriented boundary curve (i.e. its boundary curve whose orientation aligns with that of S), and let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a vector field in three dimensions such that all second partial derivatives of P, Q, and R exist and are continuous on an open region around S.

Prove the following "integration by parts" formula:

$$\iint_S (\text{grad}(\text{div}\,\mathbf{F})) \bullet d\mathbf{S} = \int_C (\,\text{curl}\,\mathbf{F}) \bullet d\mathbf{r} + \iint_S (\nabla^2\mathbf{F}) \bullet d\mathbf{S}.$$

3. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.