MATH 53 DISCUSSION SECTION ANSWERS – 4/27/23

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1. The divergence theorem

(1) **(textbook 16.9.5)** Find $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where $\mathbf{F} = xye^z\mathbf{i} + xy^2z^3\mathbf{j} - ye^z\mathbf{k}$ and S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$.

Since the surface we are working with is the surface of a box, it will have six pieces. Furthermore, our vector field is complicated. These two facts suggest that we should try using the divergence theorem.

We have $\operatorname{div} \mathbf{F} = ye^z + 2xyz^3 - ye^z = 2xyz^3$. The divergence theorem tells us we have

$$\begin{aligned}\iint\limits_S \mathbf{F} \bullet d\mathbf{S} &= \int_0^3 \int_0^2 \int_0^1 2xyz^3 dzdydx \\ &= 2\int_0^3 xdx \cdot \int_0^2 ydy \cdot \int_0^1 z^3dz \\ &= \boxed{\frac{9}{2}}.\end{aligned}$$

(2) **(textbook 16.9.31)** Suppose S and E satisfy the conditions of the divergence theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint\limits_S f\mathbf{n}dS = \iiint\limits_E \nabla f dV.$$

Note that these expressions are both *vector* quantities. We'll show these expressions are equal by looking at each component individually.

The x-component of $\iint\limits_S f\mathbf{n}dS$ is equal to $\iint\limits_S \langle f, 0, 0\rangle \cdot ndS$, which is another way of writing the flux of the vector field $\langle f, 0, 0\rangle$ through the surface S, $\iint\limits_S \langle f, 0, 0\rangle \cdot d\mathbf{S}$. We can evaluate this surface integral using the divergence theorem; we have

$$\begin{aligned}\iint\limits_S \langle f, 0, 0\rangle \cdot d\mathbf{S} &= \iiint\limits_E \operatorname{div}(\langle f, 0, 0\rangle)dV \\ &= \iiint\limits_E f_x dV.\end{aligned}$$

Similarly, we compute that the y-component of $\iint\limits_S f\mathbf{n}dS$ is $\iiint\limits_E f_y dV$ and the z-component of $\iint\limits_S f\mathbf{n}dS$ is $\iiint\limits_E f_z dV$. Since the $x-$, $y-$, and z-components of $\iint\limits_S f\mathbf{n}dS$ are $\iiint\limits_E f_x dV$, $\iiint\limits_E f_y dV$, and $\iiint\limits_E f_z dV$, respectively, and these are precisely the x-, $y-$, and z-components of $\iiint\limits_E \nabla f dV$, we have

$$\iint\limits_S f\mathbf{n}dS = \iiint\limits_E \nabla f dV$$

as desired.

2. More Stokes' theorem

(3) **(from an old exam, continuation from last Tuesday)** We define the Laplacian of $\mathbf{F}$, the vector field denoted by $\nabla^2\mathbf{F}$, as

$$\nabla^2\mathbf{F} = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R\rangle,$$

where

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is the ordinary Laplacian on scalar-valued functions. It is a fact (which you don't need to prove) that

$$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$$

for all vector fields $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ in three dimensions such that all second partial derivatives of P, Q, and R exist. Let $\mathbf{S}$ be a smooth orientable surface with an orientation chosen, let C be its smooth, positively-oriented boundary curve (i.e. its boundary curve whose orientation aligns with that of S), and let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a vector field in three dimensions such that all second partial derivatives of P, Q, and R exist and are continuous on an open region around S.

Prove the following "integration by parts" formula:

$$\iint_S (\operatorname{grad}(\operatorname{div}\mathbf{F})) \bullet d\mathbf{S} = \int_C (\operatorname{curl}\mathbf{F}) \bullet d\mathbf{r} + \iint_S (\nabla^2\mathbf{F}) \bullet d\mathbf{S}.$$

The given hypotheses are enough to apply Stokes' theorem on the vector field $\operatorname{curl}\mathbf{F}$. This tells us that

$$\int_C (\operatorname{curl}\mathbf{F}) \bullet d\mathbf{r} = \iint_S \operatorname{curl}(\operatorname{curl}\mathbf{F}) \bullet d\mathbf{S}.$$

By the fact above, we know that

$$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F},$$

so that the above becomes

$$\int_C (\operatorname{curl}\mathbf{F}) \bullet d\mathbf{r} = \iint_S \operatorname{grad}(\operatorname{div}\mathbf{F}) \bullet d\mathbf{S} - \iint_S (\nabla^2\mathbf{F}) \bullet d\mathbf{S}.$$

Rearranging these terms, we have

$$\iint_S (\operatorname{grad}(\operatorname{div}\mathbf{F})) \bullet d\mathbf{S} = \int_C (\operatorname{curl}\mathbf{F}) \bullet d\mathbf{r} + \iint_S (\nabla^2\mathbf{F}) \bullet d\mathbf{S}$$

as desired.