MATH 53 DISCUSSION SECTION PROBLEMS – 4/25/23

1. Surface Integrals of vector fields

(1) True/false practice:
  (a) Every surface is orientable.
  (b) A common interpretation for surface integrals of vector fields is as flux through the surface.

(2) **(conceptual)** Suppose $\mathbf{F}(x, y, z)$ is tangent to the surface S at all points (x, y, z) on S. What is $\iint_S \mathbf{F} \bullet d\mathbf{S}$? How might we interpret $\mathbf{F}(x, y, z)$ being tangent to S and the value of this surface integral?

(3) **(conceptual)** List some interpretations of surface integrals, both of vector fields and scalar functions.

(4) **(textbook 16.7.23)** Evaluate $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ and has upward orientation (why do we specify this?) and $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$.

(5) **(from an old quiz)** Consider the parametric surface S given by $\mathbf{r}(u, v) = \langle \cos(u + v), \sin(u + v), u\rangle$ for $0 \leq v \leq 2\pi$ and $u \in \mathbb{R}$.
  (a) What kind of curves are the grid curves of this surface for constant u and for constant v?
  (b) Sketch a picture of this surface by drawing a few grid curves $u = k$ and a few grid curves $v = k$.
  (c) Set up and evaluate somehow a double integral expression in the uv-plane that gives the total flux of the vector field $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$ through the portion of this surface with $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$ and *with the outward-pointing orientation on S.*

2. Stokes' theorem

(6) True/false practice:
  (a) Let $\mathbf{F}(x, y, z)$ be a continuous vector field whose components have continuous partial derivatives on an open region which contains the surface of the earth. Since the northern hemisphere D_N, oriented so that its normal is point out to space, and the southern hemisphere D_S, oriented so that its normal is point out to space, both have the equator oriented west-to-east (i.e. counterclockwise) as their boundary, we know
$$\iint_{D_N} (\nabla \times \mathbf{F}) \bullet d\mathbf{S} = \iint_{D_S} (\nabla \times \mathbf{F}) \bullet d\mathbf{S}$$
  by Stokes' theorem.
  (b) We can use Stokes' theorem to show that irrotational vector fields defined on all of $\mathbb{R}^3$ have line integrals that are independent of path.

(7) **(textbook 16.8.13)** By explicitly computing the line integral and surface integral, verify that Stokes' theorem holds for the vector field $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} - 2\mathbf{k}$ where S is the cone $z^2 = x^2 + y^2$, $0 \leq z \leq 4$, oriented downward.

(8) **(textbook 16.8.20)** Suppose S and C satisfy the hypotheses of Stokes' theorem (where C is the positively-oriented boundary of S), and that f and g have continuous second-order partial derivatives. Prove that
  (a) $\int_C (f\nabla g) \bullet d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \bullet d\mathbf{S}$.
  (b) $\int_C (f\nabla f) \bullet d\mathbf{r} = 0$.
  (c) $\int_C (f\nabla g + g\nabla f) \bullet d\mathbf{r} = 0$.

3. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.