MATH 53 DISCUSSION SECTION ANSWERS – 4/25/23

1. Surface Integrals of vector fields

(1) True/false practice:
  (a) False: the Möbius band is not orientable.
  (b) True.

(2) In this case, we have $\mathbf{F} \bullet \mathbf{n} = 0$, so $\iint_S \mathbf{F} \bullet d\mathbf{S} = 0$. We can interpret this as saying that all the flow is occurring along the surface, not across it.

(3) Surface integrals of scalar functions can represent mass (the surface integral of density), area (the surface integral of the function 1), etc. Surface integrals of vector fields can represent flux across the surface (where the underlying vector field represents the flow of the fluid/the force field).

(4) Because the surface is given by $z = g(x, y) = 4 - x^2 - y^2$, we use the shortcut formula:

$$\begin{aligned}
\iint_S \mathbf{F} \bullet d\mathbf{S} &= \iint_D (-Pg_x - Qg_y + R)\, dA \\
&= \iint_D (-xy(-2x) - yz(-2y) + zx)\, dA \\
&= \int_0^1 \int_0^1 \left(2x^2y + 2y^2z + zx\right) dxdy \\
&= \int_0^1 \int_0^1 \left(2x^2y + 2y^2(4 - x^2 - y^2) + (4 - x^2 - y^2)x\right) dxdy \\
&= \int_0^1 \int_0^1 \left(2x^2y + (8y^2 - 2x^2y^2 - 2y^4) + (4x - x^3 - xy^2)\right) dxdy \\
&= \int_0^1 \left[2x^3y/3 + (8xy^2 - 2x^3y^2/3 - 2xy^4) + (2x^2 - x^4/4 - x^2y^2/2)\right]_{x=0}^{x=1} dy \\
&= \int_0^1 \left(2y/3 + (8y^2 - 2y^2/3 - 2y^4) + (2 - 1/4 - y^2/2)\right) dy \\
&= \left[y^2/3 + (8y^3/3 - 2y^3/9 - 2y^5/5) + (2y - y/4 - y^3/6)\right]_{y=0}^{y=1} \\
&= 1/3 + (8/3 - 2/9 - 2/5) + (2 - 1/4 - 1/6) \\
&= \frac{713}{180}.
\end{aligned}$$

Note that the shortcut formula always assumes the upward orientation of the graph of $z = g(x, y)$; the surface integral with the downward orientation would be the negative of this.

(5) (a) The grid curves with u constant are circles. The grid curves with v constant are helices.
  (b) Picture omitted. Because $\cos^2(u + v) + \sin^2(u + v) = 1$, the surface is part of the cylinder $x^2 + y^2 = 1$.
  (c) We have

$$\begin{aligned}
\mathbf{r}_u &= \langle -\sin(u + v), \cos(u + v), 1\rangle \text{ and} \\
\mathbf{r}_v &= \langle -\sin(u + v), \cos(u + v), 0\rangle, \text{ so} \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle -\cos(u + v), -\sin(u + v), 0\rangle.
\end{aligned}$$

This is a normal vector to the surface (and it happens to have magnitude 1), but it's not the one we want: at the point $(x, y, z) = (\cos(u+v), \sin(u+v), u)$, the vector $\langle -\cos(u+v), -\sin(u+v), 0\rangle$ points inwards rather than outwards. So we set up the integral with the negative of this normal vector:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \mathbf{F}(\mathbf{r}(u,v)) \bullet \langle \cos(u+v), \sin(u+v), 0\rangle dudv \\
&= \int_0^{2\pi}\int_0^{2\pi} \langle -\sin(u+v), \cos(u+v), u\sin(u+v)\cos(u+v)\rangle \bullet \langle \cos(u+v), \sin(u+v), 0\rangle dudv \\
&= \int_0^{2\pi}\int_0^{2\pi} 0 dudv \\
&= 0.
\end{aligned}$$

This is an instance of the situation described in problem 2: the vector

$$\mathbf{F}(\mathbf{r}(u,v)) = \langle -\sin(u+v), \cos(u+v), u\sin(u+v)\cos(u+v)\rangle = \langle -y, x, xyz\rangle$$

lies in the plane spanned by

$$\begin{aligned}
\mathbf{r}_u &= \langle -\sin(u+v), \cos(u+v), 1\rangle = \langle -y, x, 1\rangle \text{ and} \\
\mathbf{r}_v &= \langle -\sin(u+v), \cos(u+v), 0\rangle = \langle -y, x, 0\rangle,
\end{aligned}$$

namely the tangent plane to the surface, so the flow is all happening along the surface and not across it.

2. Stokes' theorem

(6) True/false practice:

(a) Let $\mathbf{F}(x,y,z)$ be a continuous vector field whose components have continuous partial derivatives on an open region which contains the surface of the earth. Since the northern hemisphere D_N, oriented so that its normal is point out to space, and the southern hemisphere D_S, oriented so that its normal is point out to space, both have the equator oriented west-to-east (i.e. counterclockwise) as their boundary, we know

$$\iint_{D_N} (\nabla \times \mathbf{F}) \bullet d\mathbf{S} = \iint_{D_S} (\nabla \times \mathbf{F}) \bullet d\mathbf{S}$$

by Stokes' theorem.

False. The boundary of D_N is the equator oriented counterclockwise, but the boundary of D_S is the equator oriented *clockwise*. This means that Stokes' theorem tells us

$$\iint_{D_N} (\nabla \times \mathbf{F}) \bullet d\mathbf{S} = -\iint_{D_S} (\nabla \times \mathbf{F}) \bullet d\mathbf{S}$$

instead.

(b) We can use Stokes' theorem to show that irrotational vector fields defined on all of $\mathbb{R}^3$ have line integrals that are independent of path.

True. Given two paths C_1 and C_2 with the same starting and ending points, we can construct the closed loop C consisting of C_1 and $-C_2$, in that order. To show that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, it is enough to show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. Since the domain of $\mathbf{F}$ is all of $\mathbb{R}^3$, we can find a surface S (there are in fact many such surfaces) for which $C = \partial S$ (i.e. for which C is its positively-oriented boundary). We can apply Stokes' theorem to the curve C and find

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_S \mathbf{0} \cdot d\mathbf{S} \\
&= 0
\end{aligned}$$

since the vector field is irrotational.

(7) **(textbook 16.8.13)** By explicitly computing the line integral and surface integral, verify that Stokes' theorem holds for the vector field $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} - 2\mathbf{k}$ where S is the cone $z^2 = x^2 + y^2$, $0 \le z \le 4$, oriented downward.

The boundary curve is the circle, $x^2 + y^2 = 16$, $z = 4$, oriented clockwise. We parametrize this curve as $\mathbf{r}(t) = \langle 4\cos(-t), 4\sin(-t), 4\rangle$, $0 \le t \le 2\pi$, where the $-$ sign inside the cosine and sine comes from the fact that we are trying to get a counterclockwise orientation. We have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -4\sin(-t), 4\cos(-t), -2\rangle \cdot \langle 4\sin(-t), -4\cos(-t), 0\rangle dt \\ &= -16 \int_0^{2\pi} \sin^2(-t) + \cos^2(-t) dt \\ &= -32\pi.\end{aligned}$$

We have $\nabla \times \mathbf{F} = \langle 0, 0, 2\rangle$. Since our surface is a portion of graph of the function $z = \sqrt{x^2 + y^2}$, we can use the shortcut formula for flux through a portion of a graph. Since we want the downward orientation on the surface, we will use $\langle f_x, f_y, -1\rangle$ as the normal vector and not $\langle -f_x, -f_y, 1\rangle$ as found in the formula in the textbook, which assumes positive orientation. We have, letting D be the disk $x^2 + y^2 \le 16$ in the xy-plane,

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(0 \cdot \frac{x}{\sqrt{x^2 + y^2}} + 0 \cdot \frac{y}{\sqrt{x^2 + y^2}} - 2\right) dA \\ &= \int_0^{2\pi} \int_0^4 -2r dr d\theta \\ &= -2 \cdot \int_0^{2\pi} \int_0^4 r dr d\theta \\ &= -32\pi,\end{aligned}$$

which matches the answer we got from computing the line integral directly. (Note that in plugging into the shortcut formula, we used $\langle P, Q, R\rangle = \nabla \times \mathbf{F}$ instead of $\mathbf{F}$, because we're calculating the surface integral of the curl of $\mathbf{F}$.)

(8) **(textbook 16.8.20)** Suppose S and C satisfy the hypotheses of Stokes' theorem (where C is the positively-oriented boundary of S), and that f and g have continuous second-order partial derivatives.

(a) $\int_C (f\nabla g) \bullet d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \bullet d\mathbf{S}$.

We apply Stokes' theorem:

$$\int_C (f\nabla g) \bullet d\mathbf{r} = \iint_S \nabla \times (f\nabla g) \cdot d\mathbf{s}.$$

We compute using the definition of the curl and the product rule for partial derivatives along with Clairaut's theorem

$$\begin{aligned}\nabla \times (f\nabla g) &= \langle (fg_z)_y - (fg_y)_z, (fg_x)_z - (fg_z)_x, (fg_y)_x - (fg_x)_y\rangle \\ &= \langle f_y g_z + fg_{zy} - f_z g_y - fg_{yz}, f_z g_x + fg_{xz} - f_x g - fg_{zx}, f_x g_y + fg_{yx} - f_y g_x - fg_{xy}\rangle \\ &= \langle f_y g_z - f_z g_y, f_z g_x - f_x g_z, f_x g_y - f_y g_x\rangle \\ &= \nabla f \times \nabla g.\end{aligned}$$

Plugging this in to the right-hand side of our Stokes' theorem appplication, we have

$$\int_C (f\nabla g) \bullet d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \bullet d\mathbf{S}$$

as desired.

(b) $\int_C (f\nabla f) \bullet d\mathbf{r} = 0.$

We apply Stokes' theorem and get

$$\int_C (f\nabla f) \bullet d\mathbf{r} = \iint\limits_S \nabla \times (f\nabla f) \cdot d\mathbf{s}.$$

Using the "product rule" for curl proved in the above problem, we have $\nabla \times (f\nabla f) = \nabla f \times \nabla f = \mathbf{0}$, so we have

$$\int_C (f\nabla f) \bullet d\mathbf{r} = \iint\limits_S \mathbf{0} \cdot d\mathbf{s} = 0.$$

(c) $\int_C (f\nabla g + g\nabla f) \bullet d\mathbf{r} = 0.$

We apply the result of part (a) above, splitting the surface integral into two surface integrals first:

$$\begin{aligned} \int_C (f\nabla g + g\nabla f) \bullet d\mathbf{r} &= \int_C (f\nabla g) \bullet d\mathbf{r} + \int_C (g\nabla f) \bullet d\mathbf{r} \\ &= \iint_S (\nabla f \times \nabla g) \bullet d\mathbf{S} + \iint_S (\nabla g \times \nabla f) \bullet d\mathbf{S} \\ &= 0 \end{aligned}$$

since $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.