MATH 53 DISCUSSION SECTION ANSWERS – 4/20/23

1. Parametric surfaces and their areas

(1) This surface is the cone $x^2 + y^2 = z^2$. One way to tell this: if you divide through by s, you get the circle $\langle \cos t, \sin t, 1\rangle$, which lies flat on a plane above the origin. If you then scale this by all real numbers s, this has the effect of sliding that circle outwards to infinity, inwards to the origin, and (when $s < 0$) across the origin as well.

(2) This is given by $\mathbf{r}(u, v) = \mathbf{0} + u(\mathbf{i} - \mathbf{j}) + v(\mathbf{j} - \mathbf{k})$.

(3) There are several reasonable ways to do this. For example, we can write this in spherical coordinates as follows: $\rho = 6$, $\pi/6 \le \phi \le \pi/2$ (this corresponds to the bounds on z—draw a picture) and $0 \le \theta \le 2\pi$. Converting this into Cartesian coordinates gives the parametrization

$$\langle 6 \sin\phi \cos\theta, 6 \sin\phi \sin\theta, 6 \cos\phi\rangle,$$

with the same bounds on ϕ and θ.

(4) We have $\mathbf{r}_u = \langle 2u, 2 \sin v, \cos v\rangle = \langle 2, 0, 1\rangle$ and $\mathbf{r}_v = \langle 0, 2u \cos v, -u \sin v\rangle = \langle 0, 2, 0\rangle$, so the tangent plane is the plane passing through $\mathbf{r}(1, 0) = \langle 1, 0, 1\rangle$ and containing vectors parallel to $\langle 2, 0, 1\rangle$ and $\langle 0, 2, 0\rangle$. This in particular has normal vector given by $\langle 2, 0, 1\rangle \times \langle 0, 2, 0\rangle = \langle -2, 0, 4\rangle$, so it can be described by the equation

$$-2(x - 1) + 0(y - 0) + 4(z - 1) = 0.$$

This can be rewritten as

$$-x + 2z = 1.$$

(5) We have $\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$ and $\mathbf{r}_v = \langle -u \sin v, u \cos v, 1\rangle$, so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \cos^2 v + u \sin^2 v\rangle = \langle \sin v, -\cos v, u\rangle,$$

and thus

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1 + u^2}.$$

Thus the area is

$$\begin{aligned}\iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA &= \int_0^1 \int_0^\pi \sqrt{1 + u^2} dv du \\ &= \left(\int_0^1 \sqrt{1 + u^2} du\right)\left(\int_0^\pi 1 dv\right) \\ &= \pi \int_0^1 \sqrt{1 + u^2} du.\end{aligned}$$

This is a difficult integral, which was discussed in lecture; you have to substitute $u = \tan\theta$ and then integrate by parts and use trig identities. (If you're familiar with hyperbolic trig functions, the substitution $u = \sinh t$ leads to a slightly easier solution.) The answer turns out to be

$$\pi \int_0^1 \sqrt{1 + u^2} du = \pi \left[\frac{u\sqrt{u^2 + 1} + \log(\sqrt{u^2 + 1} + u)}{2}\right]_{u=0}^{u=1} = \pi \frac{\sqrt{2} + \log(\sqrt{2} + 1)}{2}.$$

2. Surface Integrals of functions

(6) (a) True.
(b) True:

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA.$$

(7) We can write this surface in spherical coordinates as $\rho = 2$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/2$, which leads to the parametrization

$$\mathbf{r}(\phi, \theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle.$$

This has partial derivatives

$$\begin{aligned}\mathbf{r}_\phi &= \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi\rangle,\\ \mathbf{r}_\theta &= \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0\rangle,\end{aligned}$$

which yields

$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle,\\ |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{16(\sin^4\phi + \sin^2\phi\cos^2\phi)}\\ &= \sqrt{16\sin^2\phi} = 4\sin\phi.\end{aligned}$$

Then the surface integral is given by

$$\begin{aligned}\iint_S (x^2y + y^2z)dS &= \iint_D \left((2\sin\phi\cos\theta)^2(2\sin\phi\sin\theta) + (2\sin\phi\sin\theta)^2(2\cos\phi)\right) 4\sin\phi dA\\ &= \int_0^{\pi/2}\int_0^{2\pi} (32\sin^4\phi\cos^2\theta\sin\theta + 32\sin^3\phi\sin^2\theta\cos\phi)d\phi d\theta.\end{aligned}$$

If we separate the two terms of the integrand, we can factor out the ϕ and θ parts of the functions, yielding:

$$32\left(\int_0^{\pi/2}\sin^4\phi d\phi\right)\left(\int_0^{2\pi}\cos^2\theta\sin\theta d\theta\right) + 32\left(\int_0^{\pi/2}\sin^3\phi\cos\phi d\phi\right)\left(\int_0^{2\pi}\sin^2\theta d\theta\right).$$

All of the four integrals above can be calculated with Math 1B methods. The hardest is $\int \sin^4\phi d\phi$, for which you need to use two double-angle formulas, $\sin^2\phi = \frac{1-\cos(2\phi)}{2}$ and $\cos^2\phi = \frac{1+\cos(2\phi)}{2}$. The four integrals turn out to be $3\pi/16$, 0, $1/4$, and π respectively (which in particular means you can skip the hard integral if you notice that the second one is zero), so the answer is

$$32 \cdot \frac{3\pi}{16} \cdot 0 + 32 \cdot \frac{1}{4} \cdot \pi = 8\pi.$$

In fact, if you're observant, you can skip integrating the x^2y term right from the beginning: since x^2y is an odd function of y and the given hemisphere is symmetric across the plane $y = 0$, this term will automatically integrate to 0. (The same is not true of the y^2z term because the given hemisphere is not symmetric across the plane $z = 0$.)

(8) We can parametrize S by $u = x$ and $v = y$:

$$\mathbf{r}(u, v) = \langle u, v, 4 - 2u - 2v\rangle.$$

This gives

$$\begin{aligned}\mathbf{r}_u &= \langle 1, 0, -2\rangle \text{ and}\\ \mathbf{r}_v &= \langle 0, 1, -2\rangle, \text{ so}\\ \mathbf{r}_u \times \mathbf{r}_v &= \langle 2, 2, 1\rangle \text{ and}\\ |\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{2^2 + 2^2 + 1^2} = 3.\end{aligned}$$

Thus the integral in question is

$$\iint_D x^2yz(3dA),$$

where D is the region lying below S in the plane—namely the triangle with $x \geq 0$, $y \geq 0$, and $z = 4 - 2z - 2y = 0$ (or equivalently $x + y \leq 2$). So the integral is:

$$\begin{aligned}
3\iint_D x^2yzdA &= 3\int_0^2\int_0^{2-x} x^2yzdydx \\
&= 3\int_0^2\int_0^{2-x} x^2y(4-2x-2y)dydx \\
&= 3\int_0^2\int_0^{2-x} 4x^2y - 2x^3y - 2x^2y^2dydx \\
&= 3\int_0^2 \left[2x^2y^2 - x^3y^2 - 2x^2y^3/3\right]_{y=0}^{y=2-x} dx \\
&= 3\int_0^2 \left(2x^2(2-x)^2 - x^3(2-x)^2 - 2x^2(2-x)^3/3\right) dx \\
&= 3\int_0^2 \left(2x^2(x^2-4x+4) - x^3(x^2-4x+4) - 2x^2(-x^3+6x^2-12x+8)/3\right) dx \\
&= 3\int_0^2 \left(-x^5/3 + 2x^4 - 4x^3 + 8x^2/3\right) dx \\
&= 3\left[-x^6/18 + 2x^5/5 - x^4 + 8x^3/9\right]_{x=0}^{x=2} \\
&= 3\left(-64/18 + 64/5 - 16 + 64/9\right) = \frac{16}{15}.
\end{aligned}$$