MATH 53 DISCUSSION SECTION PROBLEMS – 4/18/23

1. Curl and divergence

(1) True/false practice:
   (a) We have only defined the curl for vector fields whose domains are subsets of $\mathbb{R}^3$.
   (b) We have only defined the divergence for vector fields whose domains are subsets of $\mathbb{R}^3$.
   (c) The Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ for a function f of three variables is given by $\nabla^2 f = \operatorname{div} \operatorname{grad} f$.

(2) **(conceptual)** Give an example of a vector field, and sketch a picture near the origin, with
   (a) Positive divergence at the origin
   (b) Negative divergence at the origin
   (c) Nonzero curl at the origin

(3) **(conceptual, type issues)** Let f be a scalar function defined on $\mathbb{R}^3$ and let $\mathbf{F}$ be a vector field defined on $\mathbb{R}^3$. Assume that all functions and all components of all vector fields below have continuous second partial derivatives. For each of the following expressions, state (1) whether it makes sense to write it down, (2) what type of object it gives you, and (3) whether it is always 0 (or $\mathbf{0}$, if the object that it gives you is a vector):
   (a) $\nabla \times (\nabla \times \mathbf{F})$
   (b) $\nabla \bullet (\nabla \bullet \mathbf{F})$
   (c) $\nabla \times (\nabla \bullet \mathbf{F})$
   (d) $\nabla \times (\nabla f)$
   (e) $\nabla(\nabla \bullet \mathbf{F})$
   (f) $\nabla \bullet (\nabla f)$
   (g) **(*)** $\nabla \times (f \nabla f)$

(4) **(textbook 16.5.1)** Find the curl and divergence of $\mathbf{F}(x, y, z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$.

(5) **(textbook 16.5.3)** Find the curl and divergence of $\mathbf{F}(x, y, z) = xye^z\mathbf{i} + yze^x\mathbf{k}$.

(6) **(textbook 16.5.15)** Use curl to check whether $\mathbf{F}(x, y, z) = z\cos y\mathbf{i} + xz\sin y\mathbf{j} + x\cos y\mathbf{k}$ is conservative.

(7) **(from an old exam)** We can define the *Laplacian of a vector field* as follows. Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a vector field in three dimensions such that all second partial derivatives of P, Q, and R exist and are continuous. We define the Laplacian of $\mathbf{F}$, the vector field denoted by $\nabla^2\mathbf{F}$, as

$$\nabla^2\mathbf{F} = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R\rangle,$$

where

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is the ordinary Laplacian on scalar-valued functions.

It is a fact that

$$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$$

for all vector fields $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ in three dimensions such that all second partial derivatives of P, Q, and R exist.

Prove this fact in the case $\mathbf{F} = \langle P(x, y), Q(x, y), 0\rangle$ (i.e. the case where there is no dependence on z and the z-component of the vector field is constant at 0).

2. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked (*) are challenge problems, with problems marked (**) especially challenging problems.