MATH 53 DISCUSSION SECTION ANSWERS – 4/18/23

1. Curl and divergence

(1) (a) True, and there is no perfect analogue of curl in other numbers of dimensions. It's like a cross product; recall that curl can be written in shorthand as $\nabla \times \mathbf{F}$.
(b) The textbook only defines it on $\mathbb{R}^3$, as far as I can tell. But we saw in lecture that it makes sense on $\mathbb{R}^2$ too (or in fact on $\mathbb{R}^n$ for any n). It's like a dot product; recall that div can be written in shorthand as $\nabla \bullet \mathbf{F}$.
(c) True, this is a formula from lecture (and you can verify it directly).

(2) (a) An example is $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$; the arrows should be separating from each other.
(b) An example is $\mathbf{F}(x, y, z) = -\langle x, y, z\rangle$; the arrows should be pointing in towards each other.
(c) An example is $\mathbf{F}(x, y, z) = \langle -y, x, 0\rangle$; the arrows should be swirling around some axis.

(3) (a) If $\mathbf{F}$ is a vector field, then its curl $\nabla \times \mathbf{F}$ is too, so the curl of the curl makes sense and is a vector field. It's not always zero; for example, if $\mathbf{F} = e^x\mathbf{j}$, then you can calculate that $\nabla \times (\nabla \times \mathbf{F}) = -\mathbf{F} \neq \mathbf{0}$.
(b) This doesn't make sense: the divergence of a vector field is a function (i.e. a scalar field), so you can't take the divergence again.
(c) This also doesn't make sense: the divergence of a vector field is a function, so you can't take its curl.
(d) This makes sense: the gradient of a function is a vector field, so you can take its curl (which is a vector field). The result is always $\mathbf{0}$; this is Theorem 3 in section 16.5 of the textbook.
(e) This makes sense: the divergence $\nabla \bullet \mathbf{F}$ is a function, so you can take its gradient; the result is a vector field. It's not always zero: for example, if $\mathbf{F} = x^2\mathbf{i}$, then $\nabla \bullet \mathbf{F} = 2x$, so $\nabla(\nabla \bullet \mathbf{F}) = 2\mathbf{i}$.
(f) This also makes sense: if f is a function, then ∇f is a vector field, so its divergence makes sense and is a function. It's not always zero; in fact, it equals the Laplacian of f; i.e. the sum of its pure second partial derivatives.
(g) $\nabla \times (f\nabla f)$ This makes sense: if f is a function, then ∇f is a vector field, and $f\nabla f$ is the vector field obtained by multiplying this vector field by the function f. Then this vector field has a curl $\nabla \times (f\nabla f)$, which is another vector field.

It's less obvious whether this is always zero. In fact it is, because $f\nabla f$ is conservative:

$$f\nabla f = \langle ff_x, ff_y, ff_z\rangle = \frac{1}{2}\langle (f^2)_x, (f^2)_y, (f^2)_z\rangle = \nabla(f^2/2).$$

(4) We have $P = xy^2z^2$, $Q = x^2yz^2$, and $R = x^2y^2z$, so the curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= (2x^2yz - 2x^2yz)\mathbf{i} + (2xy^2z - 2xy^2z)\mathbf{j} + (2xyz^2 - 2xyz^2)\mathbf{k} = 0.\end{aligned}$$

The divergence is

$$\begin{aligned}\nabla \bullet \mathbf{F} &= P_x + Q_y + R_z \\ &= y^2z^2 + x^2z^2 + x^2y^2.\end{aligned}$$

As an aside, you can check that $\mathbf{F} = \nabla f$, where $f = x^2y^2z^2/2$. This explains both calculations above: the curl of a gradient is always zero, and the divergence of a gradient is the Laplacian (i.e. the sum of the pure second partial derivatives).

(5) We have $P = xye^z$, $Q = 0$, and $R = yze^x$, so the curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= ze^x\mathbf{i} + (xye^z - yze^x)\mathbf{j} + -xe^z\mathbf{k}.\end{aligned}$$

The divergence is

$$\begin{aligned}\nabla \bullet \mathbf{F} &= P_x + Q_y + R_z \\ &= ye^z + 0 + ye^x.\end{aligned}$$

(6) We have $P = z\cos y$, $Q = xz\sin y$, and $R = x\cos y$, so the curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= (-x\sin y - x\sin y)\mathbf{i} + (-\cos y - \cos y)\mathbf{j} + (z\sin y + z\sin y)\mathbf{k}.\end{aligned}$$

Since this is not the zero vector field (and $\operatorname{curl}\operatorname{grad} f = \nabla \times (\nabla f) = 0$ for all f), the given vector field is not conservative.

(7) Given that $\mathbf{F} = \langle P(x,y), Q(x,y), 0\rangle$, the formula for the curl of $\mathbf{F}$ simplifies because $R = 0$ and all partial derivatives with respect to z are zero. We have:

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\mathbf{F} &= \operatorname{curl}\left(\langle 0, 0, Q_x - P_y\rangle\right) \\ &= \langle (Q_x - P_y)_y, -(Q_x - P_y)_x, 0\rangle \\ &= \langle Q_{xy} - P_{yy}, P_{xy} - Q_{xx}, 0\rangle.\end{aligned}$$

As for the gradient of the divergence, we have:

$$\begin{aligned}\operatorname{grad}\operatorname{div}\mathbf{F} &= \operatorname{grad}(P_x + Q_y) \\ &= \langle (P_x + Q_y)_x, (P_x + Q_y)_y, 0\rangle \\ &= \langle P_{xx} + Q_{xy}, P_{xy} + Q_{yy}, 0\rangle.\end{aligned}$$

The last term is:

$$-\nabla^2\mathbf{F} = -\langle P_{xx} + P_{yy}, Q_{xx} + Q_{yy}, 0\rangle.$$

Putting everything together, we see that

$$\begin{aligned}\operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F} &= \langle P_{xx} + Q_{xy}, P_{xy} + Q_{yy}, 0\rangle - \langle P_{xx} + P_{yy}, Q_{xx} + Q_{yy}, 0\rangle \\ &= \langle Q_{xy} - P_{yy}, P_{xy} - Q_{xx}, 0\rangle \\ &= \operatorname{curl}\operatorname{curl}\mathbf{F},\end{aligned}$$

as claimed.