MATH 53 DISCUSSION SECTION PROBLEMS – 4/13/23

1. Green's theorem

(1) **(textbook 16.4.11)** Let C be the triangle from $(0,0)$ to $(0,4)$ to $(2,0)$ and then back to $(0,0)$. Let $\mathbf{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$. Find $\int_C \mathbf{F}\cdot d\mathbf{r}$.

(2) **(textbook 16.4.17)** Use Green's theorem to find the work done by the force $\mathbf{F}(x,y) = x(x+y)\mathbf{i} + xy^2\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1,0)$, along the straight line segment from there to $(0,1)$, and then back to the origin along the y-axis. (Why isn't this just zero?)

(3) **(textbook 16.4.19)** Use one of the formulas from equation (5) in section 16.4 to find the area under one arch (i.e. for $0 \le t \le 2\pi$) of the cycloid $x = t - \sin t$, $y = 1 - \cos t$ using Green's theorem.

(4) **(from an old quiz)** Consider the vector field $\mathbf{F}(x,y) = (1-y)\mathbf{i} + (x+1)\mathbf{j}$.
   (a) Sketch a picture of this vector field.
   (b) Evaluate $\int_C \mathbf{F}\bullet\mathbf{r}$, where C is the circle $x^2+y^2 = 1$, oriented counterclockwise.
   (c) Give two ways you can tell that this vector field is not conservative.

(5) **(from an old exam)** Consider the region D consisting of the set of points (x,y) such that $1 \le x^2+y^2 \le 9$.
   (a) On a picture of D, draw the *positively-oriented* boundary of D, and draw a few unit *tangent* vectors with the correct orientation on each part of the boundary.
   (b) On a second picture of D below, draw the *positively-oriented* boundary of D, and draw a few unit *normal* vectors pointing *outward from the region* D on each part of the boundary.
   (c) Letting C denote the positively-oriented boundary of D, and letting

$$\mathbf{F}(x,y) = \frac{(-2x - y(x^2+y^2))\mathbf{i} + (x(x^2+y^2) - 2y)\mathbf{j}}{(x^2+y^2)^2},$$

   find

$$\int_C \mathbf{F}\bullet d\mathbf{r}.$$

   (d) **(*)** Letting C denote the positively-oriented boundary of D, with $\mathbf{n}$ denoting the outward-pointing unit normal vectors as you drew in part (b) and with $\mathbf{F}(x,y)$ the same vector field as in part (c) above, find

$$\int_C (F\bullet\mathbf{n})ds.$$

(6) **(textbook 16.4.31)** Use Green's theorem to prove the change of variables formula in the special case where $f(x,y) = 1$. More concretely, denoting by R the region in the xy plane corresponding to the region S in the uv-plane, show that

$$\iint\limits_R dxdy = \iint\limits_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

2. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.