MATH 53 DISCUSSION SECTION ANSWERS – 4/13/23

1. Green's theorem

(1) By Green's theorem, this is the double integral of $Q_x - P_y$ over the region D inside of the triangle—except with a minus sign in front because C traces out the boundary in the negative (clockwise) orientation. So we have:

$$\begin{aligned}
\int_C \mathbf{F} \bullet d\mathbf{r} &= \int\int_D -Q_x + P_y dA \\
&= \int\int_D -(y + \cos x - x\sin x) + (\cos x - x \sin x) dA \\
&= \int\int_D -y dA \\
&= \int_0^2 \int_0^{4-2x} -y dy dx \\
&= \int_0^2 [-y^2/2]_0^{4-2x} dx \\
&= \int_0^2 (-2x^2 + 8x - 8) dx \\
&= [-2x^3/3 + 4x^2 - 8x]_0^2 \\
&= -16/3 + 16 - 16 = -16/3.
\end{aligned}$$

(2) By Green's theorem, this is the double integral of $Q_x - P_y$ over the region D inside of the triangle:

$$\begin{aligned}
\int_C \mathbf{F} \bullet d\mathbf{r} &= \int\int_D Q_x - P_y dA \\
&= \int\int_D y^2 - x dA \\
&= \int_0^1 \int_0^{1-y} (y^2 - x) dx dy \\
&= \int_0^1 [y^2 x - x^2/2]_0^{1-y} dy \\
&= \int_0^1 y^2(1-y) - (1-y)^2/2 dy \\
&= \int_0^1 (-y^3 + y^2/2 + y - 1/2) dy \\
&= [-y^4/4 + y^3/6 + y^2/2 - y/2]_0^1 \\
&= -1/4 + 1/6 + 1/2 - 1/2 = -1/12.
\end{aligned}$$

This isn't automatically zero because the force field isn't conservative. Physically, this means the force field is generally swirling in the opposite direction of the particle's motion, and resisting the motion more than it aids it.

(3) Let's use the formula $A = \oint_C x dy$, where C is the loop that first follows the straight line from $(0, 0)$ to $(2\pi, 0)$ and then follows the cycloid backwards from there ($t = 2\pi$) to the origin ($t = 0$). (We need to go backwards in order to trace the region out in the positive orientation; we will evaluate this as the negative of the line integral in the forwards direction.) Note that the line integral of $x dy$

along the x-axis is 0, because $dy = 0$ (y isn't changing). So our answer is:

$$\begin{aligned}-\int_0^{2\pi} xy'(t)dt &= -\int_0^{2\pi} (t - \sin t)\sin t dt \\ &= -\int_0^{2\pi} \left(t \sin t - \frac{1-\cos(2t)}{2}\right) dt.\end{aligned}$$

By integration by parts ($u = t$, $dv = \sin t dt$, $du = dt$, $v = -\cos t$) we have

$$\begin{aligned}\int t \sin t dt &= -t\cos t - \int -\cos t dt \\ &= -t\cos t + \sin t + C,\end{aligned}$$

so our answer is

$$\begin{aligned}-[-t\cos t + \sin t - t/2 + \sin(2t)/4]_0^{2\pi} &= -(-2\pi + 0 - 2\pi/2 + 0) + (0 + 0 - 0 + 0) \\ &= 3\pi.\end{aligned}$$

(4) To be added later.
(5) To be added later.
(6) To be added later.

2. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.