MATH 53 DISCUSSION SECTION PROBLEMS – 4/11/23

1. Line integrals and vector fields

(1) **(textbook 16.1.36)** Consider the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$.
   (a) Sketch this vector field and then some flow lines (i.e. the paths followed by particles moving with velocity given by the vector field.
   (b) If the parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? What is $\frac{dy}{dx}$ at a point along a flow line?
   (c) If a particle is at the origin at $t = 0$ moving with velocity given by this vector field, find an equation of the path it follows.

(2) **(conceptual)** Draw examples of nonzero vector fields such that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is
   (a) zero
   (b) positive
   (c) negative

   if C is the
   (a) unit circle in the plane, oriented counterclockwise
   (b) line segment from $(3, -3)$ to $(3, 3)$ in the plane, oriented from bottom to top

(3) **(textbook 16.2.39)** Find the work done by the force field

$$\mathbf{F}(x, y) = x\mathbf{i} + (y + 2)\mathbf{j}$$

in moving an object along one arch of the cycloid

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

2. The fundamental theorem of calculus for line integrals; conservative and nonconservative vector fields

(4) **(conceptual, often difficult for students)** Draw a diagram with arrows indicating implication showing the relationship between the following properties a vector field $\mathbf{F} = \langle P(x, y), Q(x, y)\rangle$ (with P and Q infinitely differentiable) might have:
   (a) $\mathbf{F}$ is conservative.
   (b) There exists a scalar-valued function $f(x, y)$ with $\mathbf{F} = \nabla f$.
   (c) $P_y = Q_x$.
   (d) $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed loops C.
   (e) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any curves C_1 and C_2 with the same starting and ending points.

(5) **(textbook 16.3.7)** Determine whether $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

(6) **(from an old quiz)** Compute the line integral

$$\int_C \langle e^y, xe^y + 2y\rangle \bullet d\mathbf{r}$$

where C is the portion of the curve $y = 2^x$ lying between $(0, 1)$ and $(2, 4)$, oriented so that it goes from left to right (i.e. so that x is increasing as t increases).

(7) **(from an old quiz)** Compute the line integral

$$\int_C \langle -y, 0\rangle \bullet d\mathbf{r},$$

where C is the positively-oriented curve in the plane consisting of the portion of the parabola $y = x^2$ between $(-1, 1)$ and $(1, 1)$ and the portion of the line $y = 1$ between $(1, 1)$ and $(-1, 1)$.

3. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked (*) are challenge problems, with problems marked (**) especially challenging problems.