MATH 53 DISCUSSION SECTION ANSWERS – 4/11/23

1. Line integrals and vector fields

(1) (a) All vectors in the vector field should point towards the right (because of the **i**), with those in the right half-plane also pointing increasingly far up and those in the left half-plane also pointing increasingly far down. The flow lines should be parabolas opening upwards.

(b) They satisfy $x'(t) = 1$, $y'(t) = x(t)$. So $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = x(t)$.

(c) The x-coordinate is just $x(t) = t$, and the y-coordinate is a solution of the differential equation $y' = x$. All solutions to this equation are given by $y = \frac{x^2}{2} + c$; the one that passes through the origin is $y = \frac{x^2}{2}$.

(2) For the unit circle oriented counterclockwise, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will be positive if **F** generally points in the counterclockwise direction, negative if **F** generally points in the clockwise direction, and zero (for example) if **F** always points straight in or straight out. For the vertical line segmented oriented from bottom to top, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will be positive if **F** generally points up, negative if **F** generally points down, and zero (for example) if **F** always points straight left or right.

(3) This is given by the line integral

$$\int_C \mathbf{F}(x, y) \bullet d\mathbf{r},$$

where C is the given curve and **r** is its parametrization. We calculate this as follows:

$$\begin{aligned}
\int_C \mathbf{F}(x, y) \bullet d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt \\
&= \int_0^{2\pi} (x\mathbf{i} + (y + 2)\mathbf{j}) \bullet ((1 - \cos t)\mathbf{i} + (\sin t)j)\, dt \\
&= \int_0^{2\pi} ((t - \sin t)\mathbf{i} + ((1 - \cos t) + 2)\mathbf{j}) \bullet ((1 - \cos t)\mathbf{i} + (\sin t)j)\, dt \\
&= \int_0^{2\pi} (t - \sin t)(1 - \cos t) + (3 - \cos t)(\sin t)dt \\
&= \int_0^{2\pi} (t - t\cos t + 2\sin t)dt
\end{aligned}$$

Integrating by parts gives $[t^2/2 - t\sin t - 3\cos t]_0^{2\pi} = 2\pi^2$.

2. The fundamental theorem of calculus for line integrals; conservative and nonconservative vector fields

(4) Let's assume the domain D is open and connected. Then items (a), (b), (d), and (e) are all equivalent, and they imply (c). If the domain is *simply* connected (i.e. has no "holes"), then all five are equivalent. References in the textbook:

- (b) is the definition of (a); see the end of section 16.1.
- (a) implies (e) by the "Independence of Path" section in 16.3.
- Theorem 3 in section 16.3 says that (d) and (e) are equivalent.
- Theorem 4 in section 16.3 says that (e) implies (a).
- Theorem 5 in section 16.3 says that (a) implies (c).
- Theorem 6 in section 16.3 says that (c) implies (a) if D is simply connected.

(5) We use the criterion of Theorems 5 and 6 in section 16.3 of the textbook: if **F** is conservative, then $P_y = Q_x$; and if $P_y = Q_x$ and the domain is simply connected, then **F** is conservative. We calculate:

$$P_y = \frac{\partial}{\partial y}(ye^x + \sin y) = e^x + \cos y,$$
$$Q_x = \frac{\partial}{\partial x}(e^x + x\cos y) = e^x + \cos y.$$

These are equal, and the domain (namely $\mathbb{R}^2$, since the vector field is defined everywhere) is simply connected, so **F** must be conservative.

This does not however tell us what the potential function $f(x,y)$ is (i.e. the function such that $\mathbf{F} = \nabla f$). To find f, let's antidifferentiate the functions $P = f_x$ and $Q = f_y$ with respect to x and y respectively:

$$f = \int f_x dx = ye^x + x\sin y + g(y),$$
$$f = \int f_y dy = ye^x + x\sin y + h(x).$$

(Note that I've used $g(y)$ and $h(x)$ as my constants of integration because any function of y would disappear when we differentiate with respect to x, and vice versa.) We can see that $f(x,y) = ye^x + x\sin y$ has the correct partial derivatives, so this is one answer. (You could also add a constant, of course.)

(6) The most basic approach to this problem would be to parametrize C by

$$\mathbf{r}(t) = \langle t, 2^t \rangle$$

for $0 \le t \le 2$, calculate that

$$\mathbf{r}'(t) = \langle 1, 2^t \ln 2 \rangle,$$

and then compute the line integral as

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^2 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt$$
$$= \int_0^2 \langle e^{2^t}, te^{2^t} + 2 \cdot 2^t \rangle \bullet \langle 1, 2^t \ln 2 \rangle dt$$
$$= \int_0^2 \left(e^{2^t} + (te^{2^t} + 2^{t+1}) \cdot 2^t \ln 2 \right) dt.$$

This could be done, but all of the 2^t's make this an unpleasant integral. Let's find a smarter way.

By calculating $P_y = Q_x = e^y$ (and the domain $\mathbb{R}^2$ is simply connected), you can tell that the vector field is conservative. You can find its potential function f by the same method as in problem 5: antidifferentiating the two components gives

$$f = \int f_x dx = xe^y + g(y) \text{ and}$$
$$f = \int f_y dy = xe^y + y^2 + h(x).$$

So $f(x,y) = xe^y + y^2$ works. (The function $g(y) = y^2$ disappears when differentiating with respect to x, but we must include it in order for f_y to be correct.) Then the fundamental theorem for line integrals tells us that

$$\int_C \mathbf{F} \bullet d\mathbf{r} = f(2,4) - f(0,1) = (2e^4 + 4^2) - (0e^1 + 1^2) = 15 + 2e^4$$

for *any* path beginning at $(0,1)$ and ending at $(2,4)$.

(7) Let's first check whether the given vector field is conservative, in the hopes of simplifying the calculation like we did in problem 6. Since $P(x, y) = -y$ and $Q(x, y) = 0$, we have

$$P_y = -1 \neq 0 = Q_x,$$

so the vector field is not conservative. This means we really have to do the integral along the specified path. (If it were conservative, then since we're integrating around a closed loop, starting and ending at $(-1, 1)$, the answer would be $f(-1, 1) - f(-1, 1) = 0$.)

Let's split the curve into two parts; we'll parametrize and integrate along each one separately. The first part, C_1, is the segment of the parabola given by $\mathbf{r}(t) = (x, y) = (t, t^2)$ with $-1 \leq t \leq 1$. The second part, C_2, is the line segment given by $\mathbf{r}(t) = (x, y) = (-t, 1)$ with $-1 \leq t \leq 1$. (I chose $-t$ instead of t so that it starts at $(1, 1)$ and ends at $(-1, 1)$. You could also parametrize it as $(t, 1)$ with $-1 \leq t \leq 1$ and put a minus sign in front of the integral.)

The integral along C_1 is given by:

$$\begin{aligned}
\int_{C_1} \mathbf{F} \bullet d\mathbf{r} &= \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt \\
&= \int_{-1}^{1} \langle -t^2, 0 \rangle \bullet \langle 1, 2t \rangle dt \\
&= \int_{-1}^{1} (-t^2 + 0) dt \\
&= [-t^3/3]_{-1}^{1} = -2/3.
\end{aligned}$$

The integral along C_2 is given by:

$$\begin{aligned}
\int_{C_2} \mathbf{F} \bullet d\mathbf{r} &= \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt \\
&= \int_{-1}^{1} \langle -1, 0 \rangle \bullet \langle -1, 0 \rangle dt \\
&= \int_{-1}^{1} dt = 2.
\end{aligned}$$

So the overall line integral is $-2/3 + 2 = 4/3$.