MATH 53 DISCUSSION SECTION ANSWERS – 3/9/23

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1. Double integrals over general regions

(1) True/false practice:
   (a) The iterated integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dydx$$

   will give us the volume of the top hemisphere of a sphere of radius 1.
   True. The graph of the function $z = \sqrt{1-x^2-y^2}$ gives the top hemisphere of the sphere of radius 1 centered at the origin. The region $-1 \leq x \leq 1$, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ is the unit disc in the xy-plane, so we are finding the volume between the top of a sphere of radius 1 and the plane slicing the sphere in half through its equator.
   (b) If we need to find a double integral over a region that is of neither type I nor type II, there is still hope.

   True. We can split our region into a number of smaller regions, each of which is of type I or type II; the integral over the whole region will be the sum of the integrals over the smaller regions.

(2) Describe the following regions in the plane as i) a type I region and ii) a type II region:
   (a) The region above the parabola $y = x^2$ and below the line $y = 1$.

   The parabola and the line intersect at the points $(-1, 1)$ and $(1, 1)$, and in the region between them the parabola is below the line. Viewed as a type I region, this region is the set of points where $-1 \leq x \leq 1$ and $x^2 \leq y \leq 1$.
   The range of possible y values is from 0 (at the point $(0, 0)$) to 1 at the top line. At any given value of y, the largest x can be is $\sqrt{y}$ and the smallest x can be is $-\sqrt{y}$, so viewed as a type II region, this region is the set of points where $0 \leq y \leq 1$ and $-\sqrt{y} \leq x \leq \sqrt{y}$.
   (b) The region between the line $y = 2x$ and the x-axis for x between 0 and 1.

   The x-axis is the line $y = 0$. Viewed as a type I region, this is the region $0 \leq x \leq 1$ and $0 \leq y \leq 2x$.
   The values of y in this region range from 0 (along the x-axis) to 2 (at the point $(1, 2)$ at the upper-right corner). For a given value of y, the smallest x can be is $\frac{y}{2}$, and the largest x can be is 1, so viewed as a type II region this region is the region $0 \leq y \leq 2$, $\frac{y}{2} \leq x \leq 1$.
   (c) The region inside the circle with radius 1 and center $(1, 2)$.

   The smallest and largest values of x are -1 and 1, and the smallest and largest values of y are -1 and 1. Viewed as a type I region, this region is $-1 \leq x \leq 1$, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$. Viewed as a type II region, this region is $-1 \leq y \leq 1$, $-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$.
   The symmetry between these expressions shouldn't be surprising, as the circle is a very symmetrical object. We will see in the Friday 3/13 lecture a better way of setting up this integral.

(3) Evaluate the iterated integral

$$\int_0^1\int_0^y xe^{y^3}\,dxdy$$

and sketch the domain of integration. What would happen if you tried to do the integral in the other order?

The domain of integration is a triangle (see the Zoom whiteboard pictures for details).
We have

$$\begin{aligned}\int_0^1 \int_0^y xe^{y^3} dxdy &= \int_0^1 e^{y^3} \int_0^y xdxdy \\ &= \int_0^1 e^{y^3} \frac{x^2}{2}|_0^y dy \\ &= \int_0^1 \frac{y^2}{2} e^{y^3} dy \\ &= \int_0^1 \frac{1}{6} e^u du \\ &= \frac{1}{6}(e-1),\end{aligned}$$

where we make the substitution $u = y^3$, $du = 3y^2 dy$.

Trying this integral in the other order, we would have $\int_0^1 \int_x^1 e^{y^3} dydx$; the function e^{y^3} does not have an elementary antiderivative we can write down, so this order is impossible, and we have to use the given order.

(4) Evaluate the double integral

$$\iint_D \frac{y}{x^2+1} dA, \quad D = \{(x,y)|0 \le x \le 4, 0 \le y \le \sqrt{x}\}.$$

The given description of the region is as a type I region, so we could set up our integral as

$$\int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1} dydx.$$

This order seems fine computationally, so we go ahead and compute:

$$\begin{aligned}\int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1} dydx &= \int_0^4 \frac{1}{x^2+1} \int_0^{\sqrt{x}} ydydx \\ &= \int_0^4 \frac{1}{x^2+1} \frac{y^2}{2}|_0^{\sqrt{x}} dx \\ &= \int_0^4 \frac{1}{2} \cdot \frac{x}{x^2+1} dx \\ &= \int_1^{17} \frac{1}{4u} du \\ &= \frac{1}{4} \ln(17),\end{aligned}$$

where we made the substitution $u = x^2+1, du = 2xdx$.

(5) Find the volume of the solid lying under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

This volume is given by the integral of $3x + 2y$ over the reigon in the xy-plane between these two parabolas. We see that the parabolas intersect at $(0,0)$ and $(1,1)$, and that between $x = 0$ and $x = 1$, $\sqrt{x} > x^2$, so we can set up our integral as follows (see the Zoom whiteboard pictures for a sketch):

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (3x+2y) dydx.$$

We have

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (3x+2y)dydx = \int_0^1 3xy + y^2\Big|_{x^2}^{\sqrt{x}} dx$$
$$= \int_0^1 (3x^{3/2} - 3x^3 + x - x^4)dx$$
$$= \frac{6}{5}x^{5/2} - \frac{3}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{5}x^5\Big|_0^1$$
$$= \frac{3}{4}.$$

(6) Give upper and lower bounds for the value of the integral

$$\iint_S \sqrt{4 - x^2y^2}dA, \quad S = \{(x,y)|x^2 + y^2 \leq 1, x \geq 0\}.$$

We use the fact that

$$Area(D)\min_D f(x,y) \leq \iint_D f(x,y)dA \leq Area(D)\min_D f(x,y).$$

Our region is one-half of a disc of area 1, so it has area $\frac{\pi}{2}$. The largest values $f(x,y) = \sqrt{4 - x^2y^2}$ takes on in region is 2, at the point $(0,0)$, while the smallest value it takes on is $\sqrt{15}2$, at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. This means we have

$$\frac{\pi\sqrt{15}}{4} \leq \iint_S \sqrt{4 - x^2y^2}dA \leq \pi.$$

These bounds are approximately 3.04 and 3.14, and the actual value for the integral is around 3.12; note that while this integral is impossible for us to compute directly, we were able to make a decent estimate. Using the properties of the function and its partial derivatives to justify that a certain kind of Riemann sum (lower-left corner, say) is guaranteed to be an underestimate always is

(7) Evaluate the double integral

$$\int_0^1 \int_{x^2}^1 e^{y^{3/2}} dydx.$$

Note (since the typsetting might not be clear) that the exponent is the function $y^{3/2}$.

We see that we are trying to integrate $e^{y^{3/2}}$ with respect to y as the inner integral here; this is a classic problem type where we will want to change the order of integration, since this function has no elementary antiderivative we can write down. The region $0 \leq x \leq 1$, $x^2 \leq y \leq 1$ can be expressed as a type II region (i.e. with the other order of integration) as $0 \leq y \leq 1$, $0 \leq x \leq \sqrt{y}$; see the Zoom whiteboard pictures for a sketch of this situation. We have

$$\int_0^1 \int_{x^2}^1 e^{y^{3/2}} dydx = \int_0^1 \int_0^{\sqrt{y}} e^{y^{3/2}} dxdy$$
$$= \int_0^1 xe^{y^{3/2}}\Big|_0^{\sqrt{y}} dy$$
$$= \int_0^1 y^{1/2}e^{y^{3/2}} dy$$
$$= \int_0^1 \frac{2}{3}e^u du$$
$$= \frac{2}{3}(e-1),$$

where we made the substitution $u = y^{3/2}, du = \frac{3}{2}y^{1/2}dy$.

When James gave this as a midterm problem in the summer of 2018, a few students tried changing the order of integration on part (b) by writing $\int_{x^2}^1 \int_0^1 e^{y^{3/2}}\,dxdy$; you always want your outermost bounds to be a pair of constants.