MATH 53 DISCUSSION SECTION SOLUTIONS – 3/7/23

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1. Lagrange multipliers

(1) True/false practice:

(a) When using Lagrange multipliers to find the maximum of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, we always get a system of linear equations in x, y, z, λ which we will immediately know how to solve.

False. We often get a nonlinear system of equations, and there's no general approach to solving these. Common tactics for solving these include trying to solve for one variable in one equation and then substitute this into another equation and adding/subtracting multiples of the first two equations to get something simpler.

(b) The geometric intuition behind the method of Lagrange multipliers is that the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, if they exist, should correspond to the points where the level surfaces of f are tangent to the constraint surface $g(x, y, z) = k$.

True. Just as ∇f is perpendicular to the level curves for a two-variable function $f(x, y)$, ∇f is perpendicular to the level surfaces of a three-variable function $f(x, y, z)$, so the Lagrange multiplier system in three variables is coming from this "tangent level surfaces" idea as well.

(2) **(textbook 14.8.5)** Given that the extreme value problem has a solution with both a maximum value and a minimum value, use Lagrange multipliers to find the extreme values of $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

Our constraint curve is $g(x, y) = 4x^2 + y^2 = 8$. We have $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 8x, 2y\rangle$. Our Lagrange multiplier system is:

$$y = 8\lambda x \tag{1}$$

$$x = 2\lambda y \tag{2}$$

$$4x^2 + y^2 = 8. \tag{3}$$

We plug equation (2) into equation (1) to get

$$y = 8\lambda(2\lambda y),$$

or, equivalently, $y = 16\lambda^2 y$. This tells us that either $y = 0$ or $16\lambda^2 = 1$.

If $y = 0$, equation (3) (the constraint) tells us that $x = \pm\sqrt{2}$. This means that $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are possible locations for the maximum and minimum subject to the constraint.

If $16\lambda^2 = 1$, we know that $\lambda = \pm\frac{1}{4}$.

If $\lambda = \frac{1}{4}$, equation (1) becomes $y = 2x$. Plugging this into equation (3), we have $4x^2 + (2x)^2 = 8$, or, equivalently, $8x^2 = 8$. This gives $x = \pm 1$. This means that $(1, 2)$ and $(-1, -2)$ are possible locations for the maximum and minimum subject to the constraint.

If $\lambda = -\frac{1}{4}$, equation (1) becomes $y = -2x$. Plugging this into equation (3), we have $4x^2 + (-2x)^2 = 8$, or, equivalently, $8x^2 = 8$. This gives $x = \pm 1$. This means that $(1, -2)$ and $(-1, 2)$ are possible locations for the maximum and minimum subject to the constraint.

Comparing the values of $f(x, y) = xy$ at these six candidate points, we see that we attain a maximum of 2 at the points $(1, 2)$ and $(-1, -2)$ and a minimum of -2 at the points $(-1, 2)$ and $(1, -2)$.

(3) **(textbook 14.8.21)** Find the extreme values of $f(x,y) = x^2+y^2+4x-4y$ on the region $x^2+y^2 \leq 9$.

This region is a closed disk of radius 3 in the plane, so it is closed and bounded, and we know the extreme values will exist. There are two places to check: any interior critical points and the boundary.

$\nabla f = \langle 2x+4, 2y-4\rangle$, so the critical points of f are the solutions to the system

$$2x + 4 = 0 \tag{4}$$
$$2y - 4 = 0. \tag{5}$$

We see that there is only one critical point for f, $(-2, 2)$. At this point $f = -8$; we will compare this to the solutions we get from the Lagrange multiplier system on the boundary.

The boundary is the level curve $x^2 + y^2 = 9$. With $g(x, y) = x^2 + y^2$, we set up our lagrange multiplier system:

$$2x + 4 = \lambda 2x \tag{6}$$
$$2y - 4 = \lambda 2y \tag{7}$$
$$x^2 + y^2 = 9. \tag{8}$$

Adding the first two equations, we get $2x + 2y = \lambda(2x + 2y)$, so either $2x + 2y = 0$ or $\lambda = 1$. If $\lambda = 1$, then equation (1) becomes $2x + 4 = 2x$, which never holds, so we can't have $\lambda = 1$. We must have $2x + 2y = 0$, i.e. that $y = -x$. Plugging this into equation (3), we get $x^2 + (-x)^2 = 9$, so that $x = \pm\frac{3\sqrt{2}}{2}$. At $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$, $f = 9 + 12\sqrt{2}$. At $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$. $f = 9 - 12\sqrt{2}$.

Comparing our three candidate points, we see that we have an absolute minimum of -8 at $(-2, 2)$ and an absolute maximum of $9 + 12\sqrt{2}$ at $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$.

(4) **((*), textbook 14.8.49)** Find the maximum value of $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1x_2\cdots x_n}$ given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = n$. Why does this imply that for all positive numbers $x_1, x_2, \ldots, x_n$,

$$\sqrt[n]{x_1x_2\cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}?$$

When does equality hold?

The Lagrange multiplier system is

$$\frac{1}{n}(x_2x_3\cdots x_n)(x_1x_2\cdots x_n)^{\frac{1}{n}-1} = \lambda \tag{9}$$
$$\frac{1}{n}(x_1x_3\cdots x_n)(x_1x_2\cdots x_n)^{\frac{1}{n}-1} = \lambda \tag{10}$$
$$\vdots = \vdots \tag{11}$$
$$\frac{1}{n}(x_1x_2\cdots x_{n-1})(x_1x_2\cdots x_n)^{\frac{1}{n}-1} = \lambda \tag{12}$$
$$x_1 + x_2 + \cdots + x_n = n. \tag{13}$$

When we solve this system (noting that the first n equations are symmetric), we find that the only solution occurs when $x_1 = x_2 = \cdots = x_n = 1$. Testing some other point (say, when all the x_i except x_n are 10^{-100} and x_n is $n - (n-1)10^{100}$, we see that other points on the constraint surface give smaller values of f,so the maximum value of $f(x_1, x_2, \ldots, x_n)$ is 1.

Given any positive numbers $x_1, x_2, \ldots, x_n$ whose sum is some positive number S, we can run the same argument and see that the maximum value of $\sqrt[n]{x_1x_2\cdots x_n}$ is $\frac{x_1+x_2+\cdots+x_n}{n}$, with equality only when all the x_i are equal.

2. Double integrals over rectangles

(5) True/false practice:

(a) Analogous to the midpoint rule for approximating integrals of functions of one variable, we have a midpoint rule for approximating double integrals.

True. We use the midpoint of each of the subrectangles we've divided the region into as the sample point.

(b) When expressing a double integral of a continuous function $f(x, y)$ over a rectangular region R of the form $\{(x, y)|a \leq x \leq b, c \leq y \leq d\}$ as an iterated integral, we do get to make at least one choice that could make our lives easier.

True. We can pick the order of integration, which can often simplify the integration.

(c) $\int_0^1 \int_0^x e^{xy} dxdy$ is a valid iterated integral to write down.

False. We can't use x as both the bound and the variable of integration for the inner integral here; the bounds on the inner integral need to look like constants from the perspective of x.

(6) **(textbook 15.1.3)** Estimate $\int\int_R xe^{-xy} dA$, where $R = [0, 2] \times [0, 1]$ by using a Riemann sum with $m = n = 2$ and using upper-right corners. Then estimate it again using the midpoint rule.

For upper-right corners, our sample points are $(1, 0.5)$, $(1, 1)$, $(2, 0.5)$, and $(2, 1)$. The area of each rectangle is 0.5. Some computation gives an approximate answer of 0.99.

For midpoints, our sample points are $(0.5, 0.25)$, $(0.5, 0.75)$, $(1.5, 0.25)$, and $(1.5, 0.75)$. The area of each rectangle is 0.5. Some computation gives an approximate answer of 1.15.

(7) **(textbook 15.1.23)** Evaluate the double integral

$$\int_0^3 \int_0^{\pi/2} t^2 \sin^3 \phi d\phi dt.$$

Writing the inner integral as $\int_0^{pi/2} (1 - \cos^2 \phi) \sin \phi d\phi$, we make the substitution $u = \cos \phi$, $du = -\sin \phi d\phi$. The inner integral becomes $\int_0^1 (1 - u^2) du$, which is $\frac{2}{3}$. The full iterated integral is thus $\int_0^3 t^2 \cdot \frac{2}{3} dt$, which is 6.

(8) **(textbook 15.1.43)** Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y - 2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.

The volume underneath the paraboloid $2 + x^2 + (y - 2)^2$ bounded by the four planes $x = 1$, $x = -1$, $y = 0$, and $y = 4$ is

$$\int_{-1}^1 \int_0^4 \left(2 + x^2 + (y - 2)^2\right) dydx,$$

which is $\frac{88}{3}$ However, the volume between the xy-plane and the plane $z = 1$ is the volume of a rectangular prism with side lengths 2, 4, and 1, which is 8. The volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y - 2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$ is thus $\frac{88}{3} - 8 = \frac{64}{3}$.

(9) **(textbook 15.1.49)** Evaluate the double integral

$$\iint_R \frac{xy}{1 + x^4} dA, \quad R = \{(x, y)| -1 \leq x \leq 1, 0 \leq y \leq 1\}.$$

We exploit symmetry: our region of integration is symmetric about the y-axis, and our integrand is odd in the x variable (i.e. $f(-x, y) = -f(x, y)$). This means that the positive contributions from the $x > 0$ portion of R are exactly cancelled by the negative contributions from the $x < 0$ portion of R, and we get a final answer of 0. Note that checking for symmetry of the region/integrand is a useful strategy for dealing with difficult or annoying integrands in cases where the region of integration is nice.

3. Notes

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.