MATH 53 DISCUSSION SECTION ANSWERS – 3/23/23

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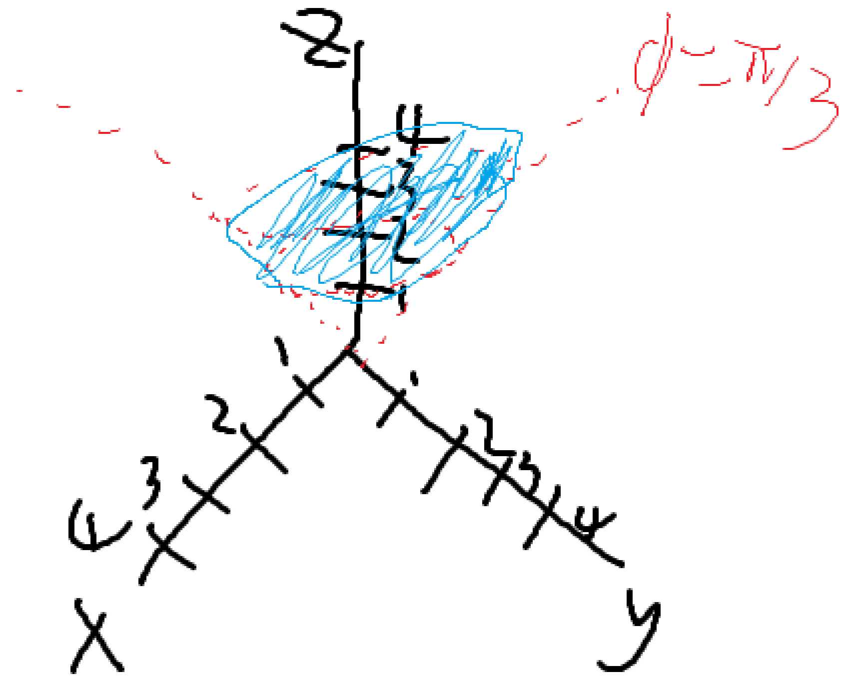
1. Triple integrals in polar coordinates

(1) Sketch the solid described by the inequalities $2 \leq \rho \leq 4$, $0 \leq \phi \leq \frac{\pi}{3}$, $0 \leq \theta \leq \pi$.

The bounds for θ tell us that our region is only on the side of the xz-plane with positive y. The bounds for ϕ tell us that our region lies within the cone with angle $\frac{\pi}{3}$, represented by the red dashed lines in the figure below. The bounds for ρ tell us that our region lies between distances of 2 and 4 from the origin. The blue region below is a rough sketch of the region:



(2) Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$.

The presence of bounds for z of the form $z = \sqrt{x^2+y^2}$ (a cone with tip at the origin) and $sz = \sqrt{2-x^2-y^2}$ (the top half of a spher of radius $\sqrt{2}$ centered at the origin indicate that spherical coordinates are a good choice for evaluating this triple integral.

The bound $z = \sqrt{x^2+y^2}$ corresponds to the surface $\rho\cos\phi = \rho\sin\phi$ (and $z \geq 0$) in spherical coordinates; this means that this surface is the surface $\phi = \frac{\pi}{4}$. Seeing that our region contains points on the positive z-axis from $(0,0,0)$ to $(0,0,\sqrt{2})$, we see that our lower bound for ϕ should be $\phi = 0$.

The bound $z = \sqrt{2-x^2-y^2}$ corresponds to the top half of the surface $\rho = \sqrt{2}$ in spherical coordinates; since our region contains the origin and all points between it and the surface of this sphere, we see that our lower bound for ρ should be $\rho = 0$.

The bounds in x and y correspond to the quarter of the circle of radius 1 centered at the origin located in the first quadrant. At $\phi = \frac{\pi}{4}$ and $\rho = \sqrt{2}$, which gives the furthest distance from the positive z-axis of our region, we see that $\sqrt{x^2+y^2} = \rho \sin\phi = 1$, so the radius of 1 gives no new information. The fact that we only wiork with the portion over the first quadrant in the xy-plane, however, tell us that our bounds for θ will be from $\theta = 0$ to $\theta = \frac{\pi}{2}$. We set up our triple integral, then, since the bounds are constants and the integrand factors as a product of functions of θ, ϕ, and ρ, can split the triple integral into a product of three single integrals:

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xydzdydx &= \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho \sin\phi\cos\theta \cdot \rho\sin\phi\sin\theta \cdot \rho^2 \sin\phi d\rho d\phi d\theta \\ &= \int_0^{\pi/4} \cos\theta\sin\theta d\theta \cdot \int_0^{\pi/4} \sin^3\phi d\phi \cdot \int_0^{\sqrt{2}} \rho^4 d\rho.\end{aligned}$$

We make the (single-variable) substitutions $u = \sin\theta$, $du = \cos\theta d\theta$, $v = \cos\phi$, $dv = -\sin\phi d\phi$, so that $\sin^3\phi d\phi = (1-\cos^2\phi)\sin\phi d\phi = -(1-v^2)dv$, and continue computing:

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xydzdydx &= \int_0^1 udu \cdot \int_1^{\sqrt{2}/2} -(1-v^2)dv \cdot \left(\frac{\rho^5}{5}\Big|_0^{\sqrt{2}} \right) \\ &= \left(\frac{u^2}{2}\Big|_0^1 \right) \cdot \left(v - \frac{v^3}{3}\Big|_{\sqrt{2}/2}^1 \right) \cdot \frac{4\sqrt{2}}{5} \\ &= \frac{1}{2} \cdot \left(\frac{2}{3} - \frac{1}{\sqrt{2}}\left(1 - \frac{1}{6}\right) \right) \cdot \frac{4\sqrt{2}}{5} \\ &= \boxed{\frac{4\sqrt{2}-5}{15}}.\end{aligned}$$

(3) Evaluate $\iiint_B (x^2+y^2+z^2)^2 dV$, where B is the ball with center the origin and radius 5.

Since we are integrating over a ball centered at the origin, spherical coordinates are a good choice for evaluating this triple integral. Converting the integrand into spherical coordinates, we are integrating ρ^4, so the integrand is also simple in spherical coordinates. We set up our triple integral, then, since the bounds are constants and the integrand factors as a product of functions of θ, ϕ, and ρ, can split the triple integral into a product of three single integrals:

$$\begin{aligned}\iiint_B (x^2+y^2+z^2)dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^5 \rho^4 \cdot \rho^2 \sin\phi d\rho d\phi d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi d\phi \int_0^5 \rho^6 d\rho \\ &= 2\pi \cdot \left(- - \cos\phi\Big|_0^{\pi} \right) \cdot \left(\frac{\rho^7}{7}\Big|_0^5 \right) \\ &= 2\pi \cdot 2 \cdot \frac{5^7}{7} = \boxed{\frac{4\pi \cdot 5^7}{7}}.\end{aligned}$$

(4) Consider the solid region E bounded by the xy-plane and the paraboloid $z = 16 - x^2 - y^2$. What is the average height of a point in E above the xy-plane?

We note that the average value of a function $f(x,y,z)$ over a region E is given by $\frac{1}{\mathrm{Vol}(E)} \iiint_E f(x,y,z)dV$, where $\mathrm{Vol}(E)$ denotes the volume of E. The height above the xy-plane of a point (x,y,z) is given by the function $f(x,z,y) = z$, and we can find the volume of E by doing $\iiint_E 1dV$.

We compute these integrals by switching to cylindrical coordinates, which work nicely for this surface since there's rotational symmetry in x and y but not z. We note that the region in the xy-plane lying below the region E is the disk $x^2+y^2 \leq 16$, and that $16 - x^2 - y^2 = 16 - r^2$. We have

$$\begin{aligned}\iiint_E 1dV &= \int_0^{2\pi}\int_0^4\int_0^{16-r^2} rdzdrd\theta \\ &= \int_0^{2\pi} d\theta \int_0^4 r\int_0^{16-r^2} dzdr \\ &= 2\pi\int_0^4 r(16-r^2)dr \\ &= 2\pi\left(8r^2-\frac{r^4}{4}\right)\Big|_0^4 \\ &= 128\pi\end{aligned}$$

and

$$\begin{aligned}\iiint_E zdV &= \int_0^{2\pi}\int_0^4\int_0^{16-r^2} zrdzdrd\theta \\ &= \int_0^{2\pi} d\theta\int_0^4 r\int_0^{16-r^2} zdzdr \\ &= 2\pi\int_0^4 r\left(\frac{z^2}{2}\right)\Big|_0^{16-r^2} dr \\ &= \pi\int_0^4 (16-r^2)^2 rdr \\ &= -\pi\int_{16}^0 \frac{u^2}{2}du \quad \text{setting } u=16-r^2,\ du=-2rdr \\ &= \pi\frac{1}{2}\frac{1}{3}u^3\Big|_0^{16} \\ &= \frac{2048\pi}{3}.\end{aligned}$$

Dividing this by the volume of E calculated above, we see that the average height is $\boxed{\frac{16}{3}}$.

(5) Using a triple integral, find the volume of the portion of the sphere of radius 2 centered at the origin lying between the cones $z=\sqrt{x^2+y^2}$ and $z=\sqrt{3x^2+3y^2}$ and above the xy-plane.

The cone $z=\sqrt{3x^2+3y^2}$ makes an angle of $\phi=\frac{\pi}{6}$ with the positive z-axis; one could see this either by drawing a cross-section of the situation or by noting that in spherical coordinates, $z=\sqrt{3x^2+3y^2}$ is the surface $\rho\cos\phi=\sqrt{3}\rho\sin\phi$, so that along this cone $\phi=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$. The cone $z=\sqrt{x^2+y^2}$ makes an angle of $\phi=\frac{\pi}{4}$ with the positive z-axis, as we've seen before (and could verify either geometrically or with a purely trigonometric argument as above). The volume of a region E in three dimensional space is given by $\iiint_E dV$, so switching to spherical coordinates (noting that the angle θ is free to range from 0 to 2π and the fact that we're working in the sphere of radius 2 centered at the origin tells us that $0\le\rho\le 2$ are our bounds on ρ), we have a volume of

$$\begin{aligned}\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^2 \rho^2\sin\phi d\rho d\phi d\theta &= \int_0^{2\pi} d\theta\int_{\pi/6}^{\pi/4}\sin\phi d\phi\int_0^2\rho^2 d\rho \\ &= 2\pi\left(-\cos\phi\Big|_{\pi/6}^{\pi/4}\right)\left(\frac{\rho^3}{3}\right)_0^2 \\ &= \boxed{\frac{8}{3}\pi(\sqrt{3}-\sqrt{2})}.\end{aligned}$$

(6) Using a triple integral, find the volume of the region lying above the cone $z = \sqrt{x^2 + y^2}$ and below the surface $z = \sqrt{4 - x^2 - y^2}$.

The volume of a region E in three dimensional space is given by $\iiint_E dV$. The presence of the surface $z = \sqrt{4 - x^2 - y^2}$, which is the top hemisphere of the sphere $x^2 + y^2 + z^2 = 4$, and the cone $z = x^2 + y^2$, suggests that spherical polar coordinates might be the easiest approach to this problem. The region above the cone $z = \sqrt{x^2 + y^2}$ corresponds to the region $0 \leq \phi \leq \frac{\pi}{4}$, the region underneath the hemisphere $z = \sqrt{4 - x^2 - y^2}$ corresponds to $0 \leq \rho \leq 2$, and the situation we are in is rotationally symmetric about the z-axis, so θ is ranging from 0 to 2π. Our volume is thus

$$\begin{aligned} \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin\phi d\rho d\phi d\theta &= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi d\phi \int_0^2 \rho^2 d\rho \\ &= 2\pi \left(-\cos\phi \Big|_0^{\pi/4} \right) \left(\frac{\rho^3}{3} \Big|_0^2 \right) \\ &= \boxed{\frac{8}{3}\pi(2 - \sqrt{2})}. \end{aligned}$$

(7) What would an analogue of spherical polar coordinates for four-dimensional space look like? What would be the "hypervolume element" (i.e. the $dV = dxdydzdw$) be for spherical polar coordinates in four dimensions?

We would want three angle variables and one length variable ρ, with ρ being the distance from the origin still. Let's call our three angle variables θ, ϕ, and ψ, with the new angle variable ψ representing the angle from the positive w-axis. We can approach defining these "hyperspherical polar coordinates" by building off of spherical polar coordinates for 3D much as we built spherical polar coordinates for 3D off of our 2D polar coordinate system.

At each point (x, y, z, w), $\rho\cos\psi$ will give the length of the "shadow" of the segment from the origin to (x, y, z, w) into the xyz-space; i.e. it will give $x^2 + y^2 + z^2$. Once we're in xyz-space, we can use the usual spherical polar coordinates. In this new system, we have

$$\begin{aligned} x &= \rho \sin\psi \sin\phi \cos\theta \\ y &= \rho \sin\psi \sin\phi \sin\theta \\ z &= \rho \sin\psi \cos\phi \\ w &- \rho \cos\psi. \end{aligned}$$

Drawing the small 4D spherical shells this coordinate system divides space into, we see that the lengths of the sides are $d\rho$, $\rho d\psi$, $\rho \sin\psi d\phi$, and $\rho \sin\psi \sin\phi d\theta$, so that $dV = \rho^3 \sin^2\psi \sin\phi$.