MATH 53 DISCUSSION SECTION ANSWERS – 3/21/23

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1. Triple integrals

(1) (a) **False.** An important application of triple integrals to three-dimensional geometry is in finding the volumes for regions, since the volume of the region E is given by $\iiint\limits_E 1dV$.

(2) Translating the description of the region as a set of points into bounds for a triple integral, we have

$$\begin{aligned}\iiint\limits_E ydV &= \int_0^3 \int_0^x \int_{x-y}^{x+y} ydzdydx \\ &= \int_0^3 \int_0^x y \cdot 2ydydx \\ &= \int_0^3 \frac{2}{3}x^3 dx \\ &= \boxed{\frac{27}{2}}.\end{aligned}$$

(3) The volume of the region is the triple integral of 1, so, as in many triple integral problems, the only difficulty in this problem is finding the bounds for the triple integral. We see that this region lies between the graphs of $z = 0$ and $z = 1 - y$ and inside a surface described in terms of x and y, so setting up our integral with z as the innermost variable is a good strategy here. The "shadow" of this region onto the xy-plane is the region between the parabola $y = x^2$ and the line $y = 1$ (the line where the two planes $z = 0$ and $y + z = 1$ intersect), and in this region the graph of $z = 1 - y$ lies above the graph of $z = 0$, so we can set up and evaluate our triple integral as follows:

$$\begin{aligned}V &= \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} 1dzdydx \\ &= \int_{-1}^1 \int_{x^2}^1 (1-y)dydx \\ &= \int_{-1}^1 \left(y - \frac{y^2}{2}\right)\Big|_{x^2}^1 dx \\ &= \int_{-1}^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx \\ &= \left(\frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10}\right)\Big|_{-1}^1 \\ &= \boxed{\frac{8}{15}}\end{aligned}$$

(4) The other five iterated integrals are:

$$\int_0^1 \int_x^1 \int_0^z f(x,y,z)dydzdx$$
$$\int_0^1 \int_0^1 \int_{\max(x,y)}^1 f(x,y,z)dzdydx$$
$$\int_0^1 \int_0^1 \int_{\max(x,y)}^1 f(x,y,z)dzdxdy$$
$$\int_0^1 \int_0^z \int_0^z f(x,y,z)dxdydz$$
$$\int_0^1 \int_0^z \int_0^z f(x,y,z)dydxdz.$$

There are a few approaches one can take here; sketching out the region in three-dimensional space (it is a relatively simple polyhedron, so if you're particularly comfortable with 3D visualization this is possible for this problem), sketching the shadows of the region on the xy-, yz-, and xz-planes, or working to manipulate the inequalities that the bounds correspond to. I'll illustrate the third strategy for a few orders.

The given bounds correspond to the inequalities

$$0 \le y \le 1$$
$$y \le z \le 1$$
$$0 \le x \le z;$$

note that the top inequality must have bounds that are constants, the second one can have bounds depending on the first variable, and the third one can have bounds depending on the first and second variables.

To rewrite the integral as an integral $dydzdx$, we can look at what the largest and smallest possible values for x are in this region. We see from the third inequality that x can be as small as 0 and (when $z = 1$, which is allowed by the second inequality) as large as 1, so we have $0 \le x \le 1$. We see from the third inequality that z must be bigger than x and from the first equation that z must be smaller than or equal to 1, so we have $x \le z \le 1$ as the bounds on z depending only on x. From the first inequality, we see that $0 \le y$, and from the second inequality, the biggest y can be at any point (x, z) is z, so we have $0 \le y \le z$. The three sets of inequalities we have found correspond exactly to the first new order above.

The third new order is a bit trickier. We see that the smallest and largest possible values for y are 0 and 1, but that at any given value of y, it is possible to have any value of x between 0 and 1, since the only variable giving an upper bound on x is z, but we can't use z to bound x because z is the innermost variable in our triple integral. For the innermost variable z, the third inequality tells us that $x \le z$, but the second inequality tells us that $y \le z$. Since we need z to be bigger than both x and y, we know that z is bigger than the maximum of x and y, which we denote as $\max(x, y)$.

Another way to give this order would be to split the triple integral into two triple integrals, one over the region in the xy-plane where $x \ge y$ and another over the region in the xy-plane where $x < y$.

(5) The average value of a function of three variables $f(x, y, z)$ over a region E is given by

$$\frac{1}{\mathrm{Vol}(E)} \iiint_E f(x,y,z)dV.$$

The volume of a cube with side length $L > 0$ is L^3. Integrating the function $f(x, y, z) = xyz$ over this cubical region and splitting the integral into a product of three single integrals since all the bounds are constants and the integrand is a product of single-variable functions of x, y, and z, we have

$$\begin{aligned}\int_0^L \int_0^L \int_0^L xyz\,dz\,dy\,dx &= \int_0^L x\,dx \int_0^L y\,dy \int_0^L z\,dz \\ &= \frac{L^2}{2} \cdot \frac{L^2}{2} \cdot \frac{L^2}{2} \\ &= \frac{L^6}{8}.\end{aligned}$$

Dividing this triple integral by the volume L^3, we have an average value of $\boxed{\frac{L^3}{8}}$ for $f(x,y,z) = xyz$ over this cube.

2. Triple integrals in cylindrical coordinates

(6) (a) **True**. For regions where you have rotational symmetry around the x or y-axes, cylindrical coordinates can be set up with (r,θ) describing the shadow on the yz-plane or xz-plane, respectively.

(7) The outer bounds in x and y are a circle of radius 2 centered at the origin in the xy-plane, and the inner bounds in z depend only on x^2+y^2, so cylindrical coordinates are a good choice for this problem. We see that the lower bound $z = \sqrt{x^2+y^2}$ is the bound $z = r$, and the upper bound remains $z = 2$, while the outer bounds become $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Setting up our integral, taking care to remember that in cylindrical coordinates, $dV = r\,dz\,dr\,d\theta$, we have

$$\begin{aligned}\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy &= \int_0^{2\pi} \int_0^2 \int_r^2 r\cos\theta \cdot z \cdot r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \cos\theta\,d\theta \int_0^2 r^2 \int_r^2 z\,dz\,dr \\ &= \int_0^{2\pi} \cos\theta\,d\theta \int_0^2 r^2 \frac{z^2}{2}\Big|_r^2 dr \\ &= 0 \cdot \int_0^2 r^2 \frac{1}{2}(4-r^2)\,dr \\ &= \boxed{0}.\end{aligned}$$

We could also notice that the region of integration in this problem is the region between the cone $z^2 = x^2+y^2$ and the plane $z = 2$, and that this region is reflection-symmetrical over the yz-plane, while the integrand xz is an odd function of x, so this integral must be 0 by symmetry.

(8) Since we are working inside a cylinder centered along the z-axis, cylindrical coordinates are a good choice for this problem. The cylinder $x^2+y^2 = 1$ corresponds to $0 \le r \le 1$, $0 \le \theta \le 2\pi$. The surface $z^2 = 4x^2+4y^2$ corresponds to the surface $z = 2r$ in cylindrical coordinates, and the plane $z = 0$ will be the lower bound for z. We have

$$\begin{aligned}\iiint_E x^2\,dV &= \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2\cos^2\theta\, r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \cos^2\theta\,d\theta \int_0^1 r^3 \int_0^{2r} dz\,dr \\ &= \frac{\cos(2\theta)+1}{2}\Big|_0^{2\pi} \cdot \int_0^1 r^3 \cdot 2r\,dr \\ &= \pi \cdot \int_0^1 2r^4\,dr \\ &= \boxed{\frac{2}{5}\pi.}\end{aligned}$$

(9) The paraboloid $z = 4x^2 + 4y^2$ intersects the plane $z = a$ when $4x^2 + 4y^2 = a$. This is a circle (in the plane $z = a$) with radius $\frac{\sqrt{a}}{2}$ centered along the z-axis. The paraboloid $z = 4x^2 + 4y^2$ is the paraboloid $z = 4r^2$ in cylindrical coordinates as well, so cylindrical coordinates are a good choice here.

To find the mass of the solid, we do the triple integral of the density over the solid. Setting this integral up in cylindrical coordinates, we have

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{\sqrt{a}/2} \int_{4r^2}^{a} K r dz dr d\theta \\ &= K \int_0^{2\pi} d\theta \int_0^{\sqrt{a}/2} (a - 4r^2) r dr \\ &= 2\pi K \cdot \left(\frac{ar^2}{2} - r^4 \right) \Big|_0^{\sqrt{a}/2} \\ &= \boxed{\frac{a^2 \pi K}{8}}. \end{aligned}$$

To find the center of mass of the solid, we can use the fact that both the region and the density function are rotationally symmetric about the z-axis, so the x and y-coordinates of the center of mass must be 0 (i.e. the center of mass must lie on the positive z-axis. To find the remaining coordinate, we find $\frac{1}{M} \iiint\limits_S z\rho(x, y, z) dV$: Setting this integral up in cylindrical coordinates, we have

$$\begin{aligned} \frac{1}{M} \iiint\limits_S z\rho(x, y, z) dV &= \frac{8}{a^2 \pi K} \int_0^{2\pi} \int_0^{\sqrt{a}/2} \int_{4r^2}^{a} K z r dz dr d\theta \\ &= \frac{8}{a^2 \pi} \int_0^{2\pi} d\theta \int_0^{\sqrt{a}/2} (\frac{a^2}{2} - 8r^4) r dr \\ &= \frac{16}{a^2} \cdot \left(\frac{a^2 r^2}{4} - \frac{4}{3} r^6 \right) \Big|_0^{\sqrt{a}/2} \\ &= \frac{2a}{3}, \end{aligned}$$

so the center of mass of the object is $\boxed{\left(0, 0, \frac{2a}{3}\right)}$.