MATH 53 DISCUSSION SECTION PROBLEMS – 3/2/23

1. Directional derivatives and the gradient vector

(1) Consider the function $f(x, y) = (x^2 + y^2) \arctan \frac{y}{x}$ defined on the half-plane $x > 0$.
   (a) Find ∇f.
   (b) At the point (x_0, y_0), find $D_{\mathbf{u}} f$, where $\mathbf{u} = \left\langle -\frac{y_0}{\sqrt{x_0^2+y_0^2}}, \frac{x_0}{\sqrt{x_0^2+y_0^2}} \right\rangle$.
   (c) Interpret your answer from part (b) in terms of polar coordinates (that is, this computation should be simpler to describe in polar coordinates).

(2) Let $f(x, y) = x^3 y^2$. Find all directions $\mathbf{u}$ such that $D_{\mathbf{u}} f(1, 1) = 2$.

(3) Find all the points on the surface $x^4 + 2x^2y^2 + y^4 - 2x^2 - 2y^2 + z^4 = 4$ where the surface has a horizontal tangent plane.

(4) **(*)** Suppose we are a computer trying to find the minimum of the function $f(x, y) = x^4 + x^2y^2 + y^4 - 2x + 4y$. Suppose we know the gradient of this function (in practice, partial derivatives are easy to at least approximate using difference quotients). We can pick a starting point, say $(2, -1)$, and try to walk towards the minimum by "walking downhill," always going in the opposite direction of ∇f; this is the idea behind the famous *method of gradient descent*. What happens if our approximation scheme is to start at a point (x_0, y_0), then go to the point (x_1, y_1) corresponding to the vector $\langle x_0, y_0 \rangle - \nabla f$, and continually repeat this process? If this process doesn't work (try it with a calculator or computer), how could we modify this strategy to work better? What are the other possible problems with this approach beyond the issue of convergence?

2. Maxima and minima of functions of two variables

(5) True/false practice:
   (a) Any continuous function $f(x, y)$ must attain an absolute maximum and minimum on the set of points (x, y) such that $x^2 + y^2 < 4$.
   (b) The set $\{(x, y) | 1 \leq x^2 + y^2 \leq 4\}$ is a closed and bounded set.

(6) **(textbook 14.7.5)** Find the local maximum and minimum values and saddle points of $f(x, y) = x - y - x^2y + xy^2$.

(7) **(textbook 14.7.33)** Find the absolute (global) maximum and minimum values of $f(x, y) = x^2 + y^2 + x^2y + 4$ on the region $D = \{(x, y) \big| |x| \leq 1, |y| \leq 1\}$. Where are they attained?

(8) **((*), if you've seen some linear algebra/Math 54)** One can show that the second directional derivative in the direction $\mathbf{u}\langle a, b \rangle$ of the function f is given by $a^2 f_{xx} + ab f_{xy} + ba f_{yx} + b^2 f_{yy}$, which is really the product $\mathbf{u} \nabla^2 f \mathbf{u}^T$ for the matrix $\nabla^2 f$ of second partial derivatives. Can you rewrite the second derivatives test in a more compact form in terms of properties of the matrix $\nabla^2 f$? This could generalize the test to higher dimensions, where the procedure as given in the book will break down. Hint to get you started: the expression $f_{xx} f_{yy} - f_{xy}^2$ is the determinant of $\nabla^2 f$.

3. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.