MATH 53 DISCUSSION SECTION ANSWERS – 3/2/23

1. Directional derivatives and the gradient vector

(1) (a) We have

$$\begin{aligned}\nabla f &= \langle f_x, f_y\rangle \\ &= \langle 2x\arctan\frac{y}{x} + (x^2+y^2)\frac{-y/x^2}{1+(y/x)^2}, 2y\arctan\frac{y}{x} + (x^2+y^2)\frac{1/x}{1+(y/x)^2}\rangle \\ &= \langle 2x\arctan\frac{y}{x} + (x^2+y^2)\frac{-y}{x^2+y^2}, 2y\arctan\frac{y}{x} + (x^2+y^2)\frac{x}{x^2+y^2}\rangle \\ &= \langle 2x\arctan\frac{y}{x} - y, 2y\arctan\frac{y}{x} + x\rangle.\end{aligned}$$

(b) We evaluate ∇f at $(x,y) = (x_0,y_0)$ and dot with $\mathbf{u}$:

$$\begin{aligned}\langle 2x_0\arctan\frac{y_0}{x_0} - y_0, 2y_0\arctan\frac{y_0}{x_0} + x_0\rangle \cdot \left\langle -\frac{y_0}{\sqrt{x_0^2+y_0^2}}, \frac{x_0}{\sqrt{x_0^2+y_0^2}}\right\rangle &= \frac{y_0^2+x_0^2}{\sqrt{x_0^2+y_0^2}} \\ &= \sqrt{x_0^2+y_0^2}.\end{aligned}$$

(c) In polar coordinates, the function is $f(r,\theta) = r^2\theta$, which has $f_r = 2r\theta$ and $f_\theta = r^2$. Imagine we walk along the circle given by $r = r_0 = \sqrt{x_0^2+y_0^2}$ and $\theta = t$. Then the tangent vector to this path at (x_0,y_0) is $r\mathrm{u} = \langle -y_0, x_0\rangle$. But we said that the function changes at a rate of $f_\theta = r^2$ while we walk along this path, so if we slow the path down by a factor of r (so that its tangent vector is u), then the resulting directional derivative of f will be $\frac{r^2}{r} = r = \sqrt{x_0^2+y_0^2}$.

(2) The gradient of f is

$$\begin{aligned}\nabla f &= \langle f_x, f_y\rangle \\ &= \langle 3x^2y^2, 2x^3y\rangle \\ &= \langle 3, 2\rangle \text{ at } (1,1).\end{aligned}$$

If $\mathbf{u} = \langle a,b\rangle$ is a unit vector, then we have

$$D_{\mathbf{u}}f(1,1) = \langle 3,2\rangle \cdot \mathbf{u} = 3a+2b.$$

So we must find all vectors $\langle a,b\rangle$ such that $a^2+b^2 = 1$ and $3a+2b = 2$. We can solve this system of equations by writing b in terms of a and solving a quadratic equation; the answer turns out to be

$$\mathbf{u} = \langle 0,1\rangle \text{ or } \left\langle \frac{12}{13}, -\frac{5}{13}\right\rangle.$$

(3) One approach would be to view the surface as a level surface of the function $f(x,y,z) = x^4+2x^2y^2+y^4-2x^2-2y^2+z^4$, and calculate ∇f. The tangent plane to the surface would then be the plane which passes through a given point and is orthogonal to ∇f. But a shortcut is to notice that the function can be rewritten in cylindrical coordinates as

$$f = (x^2+y^2)^2 - 2(x^2+y^2) + z^4 = r^4 - 2r^2 + z^4,$$

which does not depend on θ. Then the tangent plane is horizontal if and only if the tangent line of the curve $g(r,z) = r^4-2r^2+z^4 = 4$ is horizontal in the (r,z)-plane. For this, we take ∇g:

$$\nabla g = \langle 4r^3-4r, 4z^3\rangle.$$

In order for the tangent line to be horizontal, ∇g must be vertical, meaning that its first coordinate is 0 and its second coordinate is nonzero. This happens when $r = 0$ or 1 (or -1, if we let r be negative). So the tangent planes to the original surface are horizontal at points where $x = y = 0$ or $x^2 + y^2 = 1$.

(4) One possible issue is that we might overshoot the minimum when we take a step towards it. This problem can be fixed by taking many shorter steps, e.g. stepping from (x_0, y_0) to $(x_1, y_1) = \langle x_0, y_0\rangle - 0.01\nabla f$.

Another possible issue is that we might approach a local minimum that isn't a global minimum. Taking shorter steps will only make this problem worse (it's harder to "escape" a local minimum with short steps), but we can try to fix this problem by trying many different starting points, in the hopes that at least one of them will approach the absolute minimum. (You can think of the starting points as being like raindrops: they land in many different places and all run downhill, and some of them will get stuck in lakes, but others will make it out to the sea.)

2. Maxima and minima of functions of two variables

(5) (a) False: this set is not closed. If "$<$" were replaced by "$\leq$", then the set would be closed and the statement would be true.
(b) True. It's an annulus consisting of a circle of radius 2 with a circle of radius 1 removed, but including both the inner and outer boundaries. So it's contained in a finite disk, and it contains all its boundary points.

(6) We start by calculating all the first and second partial derivatives of f:

$$\begin{aligned} f_x &= 1 - 2xy + y^2, \\ f_y &= -1 - x^2 + 2xy, \\ f_{xx} &= -2y, \\ f_{xy} &= -2x + 2y, \\ f_{yy} &= 2x. \end{aligned}$$

A point (x, y) is a critical point of f if $f_x = f_y = 0$. If this happens, then

$$f_x + f_y = y^2 - x^2 = 0,$$

so we must have $y = x$ or $y = -x$. On the line $y = x$, f_x simplifies to $1 - x^2$, so $x = y = \pm 1$ are the only critical points here. On the line $y = -x$, f_x simplifies to $1 + 3x^2$, which is never 0. So the only critical points are $(1, 1)$ and $(-1, -1)$.

At $(1, 1)$, we have

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(2) - 0^2 = -4 < 0,$$

so the point is a saddle point. At $(-1, -1)$, we have

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(-2) - 0^2 = -4 < 0,$$

so this is also a saddle point.

(7) We first calculate the partial derivatives of f, in order to find the critical points of f on the interior of D:

$$\begin{aligned} f_x &= 2x + 2xy = 2x(1 + y), \\ f_y &= 2y + x^2. \end{aligned}$$

In order for f_x to equal 0, we must have either $x = 0$ or $y = -1$—but $y = -1$ can't happen in the interior of D (only on the boundary). So in order to find a critical point inside of D, we need $x = 0$, and also $y = 0$ because $f_y = 2y + 0 = 0$. Thus the only critical point inside of D is $(0, 0)$, where $f(0, 0) = 4$. We can check that this point is a local minimum from the second derivative test,[1] but this isn't actually necessary for the problem—we just need to know that $(0, 0)$ is the only interior point which has a chance of being a (local or global) minimum or maximum.

[1] It's either a local minimum or a local maximum because $f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 2 - 0^2 = 4 > 0$, and we can distinguish between local minima and maxima by looking at either f_{xx} or f_{yy} individually.

Next, we look at the boundary of D, which consists of the four line segments given by

$$x = \pm 1, -1 \leq y \leq 1,$$
$$y = \pm 1, -1 \leq x \leq 1.$$

On the lines $x = 1$ and $x = -1$, the function is given by

$$f(\pm 1, y) = y^2 + y + 5 = (y + 1/2)^2 + \frac{19}{4},$$

which has minimum value $\frac{19}{4} = 4.75$ at $y = -1/2$, and which has maximum value 7 at the endpoint $y = 1$. On the line $y = 1$, the function is given by

$$f(x, 1) = 2x^2 + 5,$$

which has minimum value 5 at $x = 0$ and maximum value 7 at the endpoints $x = \pm 1$. Finally, on the line $y = -1$, the function is constant, given by $f(x, -1) = 5$. So in summary, the smallest value we found was $f(0, 0) = 4$ and the largest was $f(1, 1) = f(-1, 1) = 7$, so these are the absolute minimum and maximum values of f on D.

(8) The simplest way to write the second derivatives test is to say that f has a local minimum or maximum at a given point if $\det(\nabla^2 f) > 0$ and a saddle point if $\det(\nabla^2 f) < 0$. (The test is inconclusive if $\det(\nabla^2 f) = 0$.) But this doesn't generalize well to higher dimensions. A better description, which follows pretty much directly from the given formula for the second directional derivative, is this: f has a local minimum if the matrix $\nabla^2 f$ is positive-definite (meaning that $\mathbf{u}(\nabla^2 f)\mathbf{u}^T$ is positive for all vectors $\mathbf{u} \neq 0$), a local maximum if it is negative-definite ($\mathbf{u}(\nabla^2 f)\mathbf{u}^T$ is negative for $\mathbf{u} \neq 0$), and a saddle point if it is indefinite ($\mathbf{u}(\nabla^2 f)\mathbf{u}^T$ is sometimes positive and sometimes negative). (The test is inconclusive if the matrix is semidefinite, meaning that the expression is sometimes zero but never positive or never negative.)

How this generalizes to higher dimensions: in any number of dimensions, the matrix $\nabla^2 f$ of second partial derivatives is a symmetric matrix by Clairaut's theorem. Then the spectral theorem tells us that $\nabla^2 f$ can be diagonalized. Diagonalizing this matrix corresponds to changing coordinates in $\mathbb{R}^n$ to force $\nabla^2 f$ to be a diagonal matrix; that is, to force all mixed partial derivatives of f to be zero. In this coordinate system, f may be concave up or down in each coordinate direction, but there will be no special second-order effects involving interactions between two different coordinate directions. Then it's easy to tell whether the given point is a local maximum or local minimum. For example, if

$$f_{xx}, f_{yy}, f_{zz} > 0,$$

then the point is a local minimum, because f increases in both directions as we change any of the coordinates. Similarly, if

$$f_{xx}, f_{yy}, f_{zz} < 0,$$

then the point is a local maximum. If some of the second partial derivatives are positive and others are negative, then the point is some kind of higher-dimensional analogue of a saddle point.

How this relates to the determinant in the two-variable case: when we diagonalized the matrix $\nabla^2 f$, its determinant didn't change. In the modified coordinate system, this determinant is just

$$\det \begin{pmatrix} f_{xx} & 0 \\ 0 & f_{yy} \end{pmatrix} = f_{xx} f_{yy},$$

since we forced f_{xy} to be 0. Thus the determinant tells whether f_{xx} and f_{yy} (the eigenvalues of the matrix!) have the same sign or opposite signs. But what we really care about is whether they're both positive or both negative (or neither). In higher dimensions, the determinant isn't enough to detect this, because for example the determinant of

$$\begin{pmatrix} f_{xx} & 0 & 0 \\ 0 & f_{yy} & 0 \\ 0 & 0 & f_{zz} \end{pmatrix}$$

is positive if either none or exactly two of f_{xx}, f_{yy}, f_{zz} are negative, and negative if one or all of them are negative. (Assuming that none of them are zero.)