MATH 53 DISCUSSION SECTION ANSWERS – 3/16/23

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1. Applications of double integrals

(1) The mass of the lamina is given by $M = \iint_D \rho(x,y)dA$. It's best to set this integral up in Cartesian coordinates as a type I region. The curves $y = 0$ and $y = 1 - x^2$ intersect at the points $(-1, 0)$ and $(1, 0)$, so we can set up and evaluate the double integral as follows:

$$\begin{aligned} M &= \int_{-1}^{1}\int_{0}^{1-x^2} ky\,dy\,dx \\ &= k\int_{-1}^{1}\left(\frac{y^2}{2}\Big|_0^{1-x^2}\right)dx \\ &= \frac{k}{2}\int_{-1}^{1}(1-2x^2+x^4)dx \\ &= \frac{k}{2}\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\Big|_{-1}^{1}\right) \\ &= \boxed{\frac{8}{15}k}. \end{aligned}$$

To find the center of mass, we can use that our region is symmetric over the y-axis and the density function is an even function with respect to the x-variable, so by symmetry the x-coordiante of the center of mass is 0. The y-coordinate is given by the following double integral expression:

$$\begin{aligned} \frac{1}{M}\iint_D y\rho(x,y)dA &= \frac{15}{8k}\int_{-1}^{1}\int_{0}^{1-x^2} ky^2\,dy\,dx \\ &= \frac{15}{8}\int_{-1}^{1}\frac{y^3}{3}\Big|_0^{1-x^2}dx \\ &= \frac{5}{8}\int_{-1}^{1}(1-3x^2+3x^4-x^6)dx \\ &= \frac{5}{8}\left(x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7\right)\Big|_{-1}^{1} \\ &= \frac{4}{7}, \end{aligned}$$

so the center of mass is $\boxed{\left(0, \frac{4}{7}\right)}$.

(2) We let H, J, and X be the random variables representing the times that Holly, James, and Xiaohan, respectively, arrive. The joint density function for the three random variables, which are independent (this will likely be specified more explicitly on an exam), is $f(h,j,x) = e^{-h-j-x}$. To describe the region whose probability we want to find, we can consider the following cases:

Case 1: all three arrive before 1pm. This is the region $H \leq 1, J \leq 1, X \leq 1$, which is the unit cube $[0,1]^3$ in hjx-space. The probability of this happening is

$$\begin{aligned}\int_0^1 \int_0^1 \int_0^1 e^{-h-j-x} dx dj dh &= \int_0^1 e^{-h} dh \int_0^1 e^{-j} dj \int_0^1 e^{-x} dx \\ &= \left(1 - \frac{1}{e}\right)^3.\end{aligned}$$

Case 2: Holly and James arrive before 1pm, but Xiaohan does not. This is the region $H \leq 1, H \leq 1, X \geq 1$. The probability of this happening is

$$\begin{aligned}\int_0^1 \int_0^1 \int_1^\infty e^{-h-j-x} dx dj dh &= \int_0^1 e^{-h} dh \int_0^1 e^{-j} dj \int_1^\infty e^{-x} dx \\ &= \left(1 - \frac{1}{e}\right)^2 \frac{1}{e}\end{aligned}$$

Case 3 (Holly and Xiaohan arrive before 1pm, but James does not) and Case 4 (James and Xiaohan arrive before 1pm, but Holly does not) can both be computed using the same approach as Case 2.

Since our four cases are disjoint (there is no overlap between them), the total probability of tea happening is

$$\left(1 - \frac{1}{e}\right)^3 + 3\left(1 - \frac{1}{e}\right)^2 \frac{1}{e} = \boxed{\left(1 - \frac{1}{e}\right)^2 \left(1 + \frac{2}{e}\right)}.$$

For the double integral version, you could consider the three cases of Holly and Xiaohan both arriving before 1pm, Holly arriving before 1pm but Xiaohan arriving after 1pm, and Holly arriving after 1pm with Xiaohan arriving before 1pm, or you could try to use *complementary probability* and find the probability that they *don't* meet for tea, which is easier to compute in this two-person case, and subtract that from 1 to get the probability that they do meet.

(3) Since we are looking at the part of the surface over an annulus (i.e. between two circles) in the plane, switching to polar coordinates is a good idea here. Letting D be our region, we have that the surface area is

$$\iint_D \sqrt{1 + (-2x)^2 + (2y)^2} dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2 \cos^2\theta + 4r^2 \sin^2\theta} r dr d\theta.$$

We use the fact that $\cos^2\theta + \sin^2\theta = 1$, then make the substitution $u = 1 + 4r^2$, $du = 8rdr$ to find

$$\begin{aligned}SA &= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2 \cos^2\theta + 4r^2 \sin^2\theta} r dr d\theta \\ &= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_5^{17} \frac{1}{8}\sqrt{u} du \\ &= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_5^{17} \\ &= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right).\end{aligned}$$

(4) This triangle is given by the portion of the graph of the function $z = 4 - x - y$ located over the triangular region in the plane with boundary points at $(0,0)$, $(4,0)$, and $(0,4)$. We have that $\frac{\partial z}{\partial x} = -1$

and $\frac{\partial z}{\partial y} = -1$, so using our formula for surface area, we have

$$\begin{aligned} SA &= \int_0^4 \int_0^{4-x} \sqrt{1+(-1)^2+(-1)^2}\,dydx \\ &= \int_0^4 \sqrt{3}y\Big|_0^{4-x} dx \\ &= \sqrt{3}\int_0^4 (4-x)dx \\ &= \sqrt{3}\left(4x - \frac{x^2}{2}\right)\Big|_0^4 \\ &= \boxed{8\sqrt{3}}. \end{aligned}$$

Note that once we saw the integrand was a constant, we could instead have computed the double integral by using the fact that the double integral of the constant function $f(x, y) = C$ over a region D is C times the area of D. Of course, for most triangles in three-dimensional space, it won't be easy to guess how to represent them as $z = f(x, y)$, and the work required to figure that out (e.g. by taking the cross product of two of the vectors representing the sides) would already be most of the way to finding the area directly without using a double integral.

2. Notes

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.