MATH 53 DISCUSSION SECTION PROBLEMS – 3/14/23

1. Techniques for double integrals

These problems are not necessarily all from section 15.3; part of the difficulty of this chapter is picking which technique to use.

(1) True/false practice:
   (a)
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)dydx = \int_0^{2\pi}\int_0^1 f(x,y)drd\theta$$
   since both expressions give the integral of f over the interior of the unit circle.
   (b) Just as we figured out a way to transform double integrals from Cartesian coordinates to polar coordinates, we should in principle be able to figure out a way to transform double integrals from Cartesian coordinates to parametric coordinates.
   (c) We can think of changing from Cartesian coordinates to polar coordinates as a kind of "integration by substitution" where we make the substitutions $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$ (and appropriately change our bounds).

(2) **(textbook 15.3.19)** Find the volume under the paraboloid $z = x^2+y^2$ and above the disk $x^2+y^2 \leq 25$ in the xy-plane.

(3) **(from an old exam)** Evaluate the double integral
$$\int_0^1\int_1^e \frac{x}{y}dydx.$$

(4) **(from an old quiz)** Evaluate the double integral
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\cos(x^2+y^2)dydx.$$

(5) **(from an old exam)** Evaluate the double integral
$$\int_0^1\int_{\sqrt{x}}^1 e^{y^3}dydx.$$

(6) **(from an old exam)** Consider a bowl whose inner surface is given by the portion of the graph of $f(x,y) = 1-(1-x^2-y^2)^{1/3}$ lying below the plane $z = 2$. Find the volume of the bowl. For an added challenge, can you make a tricky geometric argument to minimize the amount of computation you need to do here?

2. Applications of double integrals

(7) **(textbook 15.4.13)** A lamina is the region between $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ and above the x-axis. Find the center of mass if the density at any point is proportional to the distance to the origin.

(8) **(textbook 15.4.27)** The joint density function for the random variables X and Y is given by $f(x,y) = Cx(1+y)$ for $0 \leq x \leq 1$, $0 \leq y \leq 2$, and 0 otherwise.
   (a) Find the value of the constant C.
   (b) Find $P(X \leq 1, Y \leq 1)$.
   (c) Find $P(X+Y \leq 1)$.

(9) **(*)** Suppose we have a joint density function $f(x,y)$ for two random variables X and Y.
   (a) How do we find the probability density function for the random variable $X+Y$?
   (b) How do we find the probability density function for the random variable Y given that we know $X = c$ for some real number c?

(c) Consider the joint density function of two independent normal random variables of mean 1 and standard deviation 1, and the joint density function of two *Pareto random variables* with PDF $f(x) = 0$ for $x \leq 1$, $f(x) = \frac{1}{x^2}$ for $x > 1$. For both of these situations, find the probability density functions for X and Y given that $X + Y = 4$.

3. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked (*) are challenge problems, with problems marked (**) especially challenging problems.