MATH 53 DISCUSSION SECTION SOLUTIONS – 3/14/23

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1. Techniques for double integrals

These problems are not necessarily all from section 15.3; part of the difficulty of this chapter is picking which technique to use.

(1) True/false practice:

(a)
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)dydx = \int_0^{2\pi}\int_0^1 f(x,y)drd\theta$$
since both expressions give the integral of f over the interior of the unit circle.

False. The integral in polar coordinates has the correct bounds, but the area element $dydx$ should be replaced with $rdrd\theta$, not $drd\theta$.

(b) Just as we figured out a way to transform double integrals from Cartesian coordinates to polar coordinates, we should in principle be able to figure out a way to transform double integrals from Cartesian coordinates to parametric coordinates.

False. With parametric curves, we had one variable, t, tracing out our curve, and we can't describe a two-dimensional region with a single parameter. In chapter 16 with parametric surfaces, we will learn ways to describe two-dimensional regions like the surface of a sphere with two parameters.

(c) We can think of changing from Cartesian coordinates to polar coordinates as a kind of "integration by substitution" where we make the substitutions $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$ (and appropriately change our bounds).

True. This is a special case of the more general concept of *change of variables* for multiple integrals; see section 15.9 of the textbook.

(2) **(textbook 15.3.19)** Find the volume under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 25$ in the xy-plane.

The presence of a disk centered at the origin as our region of integration and an integrand that depends purely on $x^2 + y^2$ suggests that polar coordinates are a good approach to this problem. We use the fact that the volume under the graph of $z = f(x,y)$ over a region D is $\iint_D f(x,y)dA$ and express D in polar coordinates. We have

$$\begin{aligned}\int_0^{2\pi}\int_0^5 r^2\cdot rdrd\theta &= \int_0^{2\pi} d\theta\int_0^5 r^3dr\\ &= 2\pi\cdot\left(\frac{r^4}{4}\Big|_0^5\right)\\ &= \boxed{\frac{625\pi}{2}}.\end{aligned}$$

(3) **(from an old exam)** Evaluate the double integral

$$\int_0^1 \int_1^e \frac{x}{y} dydx.$$

We see that both the bounds in x and the bounds in y are pairs of constants, and that the integrand $\frac{x}{y}$ factors as the product of a function of x (x) and a function of y ($\frac{1}{y}$). We can thus split the integral up as the product of two single integrals:

$$\begin{aligned}\int_0^1 \int_1^e \frac{x}{y} dydx &= \left(\int_0^1 xdx\right)\left(\int_1^e \frac{dy}{y}\right)\\ &= \left(\frac{x^2}{2}\Big|_0^1\right)\left(\ln y\Big|_1^e\right)\\ &= \left(\frac{1}{2} - 0\right)(\ln e - \ln 1)\\ &= \frac{1}{2}.\end{aligned}$$

(4) **(from an old quiz)** Evaluate the double integral

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\cos(x^2+y^2) dydx.$$

The presence of $\sqrt{1-x^2}$ terms here, as well as an integrand given in terms of x^2+y^2, suggest switching to polar coordinates could be a good idea. We recognize the region of integration as a disk of radius 1 centered at the origin; in polar coordinates, this region is $0 \le \theta \le 2\pi$, $0 \le r \le 1$. We change to polar coordinates, recognizing that $x^2+y^2 = r^2$ and that we need to replace $dydx$ with $rdrd\theta$ and making the substitution $u = r^2$, $du = 2rdr$:

$$\begin{aligned}\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\cos(x^2+y^2) dydx &= \int_0^{2\pi} \int_0^1 2\cos(r^2) rdrd\theta\\ &= \int_0^{2\pi} d\theta \int_0^1 2r\cos(r^2)dr\\ &= \int_0^{2\pi} d\theta \int_0^1 \cos u du\\ &= 2\pi \sin u\Big|_0^1\\ &= 2\pi\sin(1).\end{aligned}$$

(5) **(from an old exam)** Evaluate the double integral

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydx.$$

The e^{y^3} term looks impossible to antidifferentiate directly, so we change the order of integration using Fubini's theorem. The region $0 \le x \le 1$, $\sqrt{x} \le y \le 1$ can be viewed as a type II region by viewing it as $0 \le y \le 1$, $0 \le x \le y^2$ (as can be seen either algebraically by rearranging the two inequalities or by sketching a picture of the region). In this order, we are able to evaluate the integral in y by substitution, letting $u = y^3$, $du = 3y^2 dy$:

$$\begin{aligned}\int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydx &= \int_0^1 \int_0^{y^2} e^{y^3} dxdy \\ &= \int_0^1 y^2 e^{y^3} dy \\ &= \frac{1}{3}\int_0^1 e^u du \\ &= \boxed{\frac{e-1}{3}}.\end{aligned}$$

(6) **(from an old exam)** Consider a bowl whose inner surface is given by the portion of the graph of $f(x, y) = 1 - (1 - x^2 - y^2)^{1/3}$ lying below the plane $z = 2$. Find the volume of the bowl. For an added challenge, can you make a tricky geometric argument to minimize the amount of computation you need to do here?

There are two general approaches we can take here: set up a triple integral of 1 or set up a double integral of the difference between $z = 2$ and $z = 1-(1-x^2-y^2)^{1/3}$ (we want this difference because the volume *inside the bowl* has to lie above the surface of the bowl; liquid below the inner surface of a bowl would fall out).

Taking the double integral approach, we see that the radial symmetry of the situation (the fact that z depends only on $x^2 + y^2 = r^2$ suggests that polar coordinates might be a good choice for how to tackle this problem. We know that z is ranging from $1 - (1 - x^2 - y^2)^{1/3}$ up to $z = 2$, so the integrand will be $2 - \left(1 - (1 - r^2)^{1/3}\right) = 1 + (1 - r^2)^{1/3})$, and that we want all angles from $\theta = 0$ to $\theta = 2\pi$, since we have complete rotational symmetry.

The only difficulty remaining is to find what the bounds on r should be. To do this, we figure out where the surface $z = 1-(1-x^2-y^2)^{1/3}$ intersects the plane $z = 2$. Equating both expressions, we have

$$\begin{aligned}1 - (1 - x^2 - y^2)^{1/3} &= 2 \\ -(1 - x^2 - y^2)^{1/3} &= 1 \\ -1 + x^2 + y^2 &= 1 \\ x^2 + y^2 &= 2,\end{aligned}$$

so the plane $z = 2$ and the surface $z = 1-(1-x^2-y^2)^{1/3}$ intersect in the curve $x^2+y^2 = 2$, $z = 2$. Since we want everything inside the bowl, and since $x^2 + y^2 = 2$, we want all values of r between 0 and $\sqrt{2}$. Our double integral giving the volume of the bowl is then

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \left(1 + (1 - r^2)^{1/3}\right) rdrd\theta,$$

remembering that dV in cylindrical coordinates is $rdzdrd\theta$.

We evaluate the double integral we set up above:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\sqrt{2}}\left(1+(1-r^2)^{1/3}\right)rdrd\theta &= \left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\sqrt{2}}\left(1+(1-r^2)^{1/3}\right)rdr\right)\\ &= 2\pi\int_0^{\sqrt{2}}\left(1+(1-r^2)^{1/3}\right)rdr\\ &= 2\pi\int_0^{\sqrt{2}}rdr+2\pi\int_0^{\sqrt{2}}(1-r^2)^{1/3}rdr\\ &= 2\pi\frac{r^2}{2}\Big|_0^{\sqrt{2}}+2\pi\int_1^{-1}-\frac{1}{2}u^{1/3}du\\ &= 2\pi(1-0)+0\\ &= 2\pi,\end{aligned}$$

where we make the substitution $u = 1 - r^2, du = -2rdr$ and use the fact that $-\frac{1}{2}u^{1/3}$ is an odd function, so its integral over the domain $[-1, 1]$, which is symmetric about the origin, is zero (we could also have evaluated the integral explicitly using the power rule and gotten zero).

The answer of 2π may be surprising, but it turns out that the symmetry of the function $\sqrt[3]{u}$ means that the volume inside the bowl and outside the cylinder $r = 1$ above the height $z = 1$ and the volume outside the bowl and inside the cylinder $r = 1$ below the height $z = 1$ exactly cancel out, so that the volume of a cylinder with base radius 1 and height 2 is equal to the volume of this bowl. Try sketching these surfaces if you're confused.

2. Applications of double integrals

(7) **(textbook 15.4.13)** A lamina is the region between $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ and above the x-axis. Find the center of mass if the density at any point is proportional to the distance to the origin.

Recall that the mass of a lamina in the shape of the region D with density $\rho(x, y)$ at each point (x, y) in D is $M = \iint_D \rho(x,y)dA$ and that the x- and y-coordinates of the center of mass of that lamina are $\frac{1}{M}\iint_D x\rho(x,y)dA$ and $\frac{1}{M}\iint_D y\rho(x,y)dA$, respectively.

The region being between two semicircles centered at the origin suggests that polar coordinates might be a good approach to these double integrals. The density function is $\rho(x, y) = k\sqrt{x^2+y^2}$ for some constant $k > 0$, so the density function is also exasy to describe in polar coordinates. We set up and evaluate the integral for the mass, noting that since we are only working above the x-axis, our bounds for θ will be $0 \le \theta \le \pi$:

$$\begin{aligned}M &= \iint_D \rho(x,y)dA\\ &= \int_0^{\pi}\int_1^2 kr\cdot rdrd\theta\\ &= \int_0^{\pi}d\theta\int_1^2 kr^2dr\\ &= k\pi\cdot\frac{r^3}{3}\Big|_1^2\\ &= \frac{7k\pi}{3}.\end{aligned}$$

We see that our density is an even function in x and our lamina has reflection symmetry over the y-axis, so we can quickly see that the x-coordinate of the center of mass is 0 (be on the lookout for such simplifying symmetries on center-of-mass problems, as they can reduce the amount of computation you need to do considerably).

To find the y-coordinate of the center of mass, we have the following expression:

$$\frac{1}{M}\iint_D y\rho(x,y)dA = \frac{3}{7k\pi}\int_0^\pi \int_1^2 r\sin\theta \cdot kr \cdot rdrd\theta$$
$$= \frac{3}{7k\pi}\int_0^\pi \sin\theta d\theta \int_1^2 kr^3 dr$$
$$= \frac{3}{7\pi}\cdot 2 \cdot \frac{15}{4} = \frac{45}{14\pi}.$$

So the center of mass of the lamina is $\boxed{\left(0, \frac{45}{14\pi}\right)}$.

(8) **(textbook 15.4.27)** The joint density function for the random variables X and Y is given by $f(x,y) = Cx(1+y)$ for $0 \le x \le 1$, $0 \le y \le 2$, and 0 otherwise.

(a) Find the value of the constant C.

For $f(x,y)$ to be a probability density function, we need the double integral of $f(x,y)$ over the whole plane to be equal to 1 (the function is already piecewise-continuous, and, provided C is greater than 0, nonnegative everywhere). We have

$$1 = \int_0^1 \int_0^2 Cx(1+y)dydx$$
$$= C\int_0^1 xdx \int_0^2 (1+y)dy$$
$$= C \cdot \frac{1}{2} \cdot \left(\frac{(1+y)^2}{2}\Big|_0^2 \right)$$
$$= 2C,$$

so that we need $C = \frac{1}{2}$.

(b) Find $P(X \le 1, Y \le 1)$.

The probability of (X,Y) lying in some region D is given by $\iint_D f(x,y)dA$, where $f(x,y)$ is the joint density function for X and Y. The region D where both X and Y are less than or equal to 1 is the square $0 \le x \le 1$, $0 \le y \le 1$. We have, using the value of C found above,

$$P(X \le 1, Y \le 1) = \int_0^1 \int_0^1 \frac{1}{2}x(1+y)dydx$$
$$= \frac{1}{2}\int_0^1 xdx \int_0^1 (1+y)dy$$
$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}$$
$$= \boxed{\frac{3}{8}}.$$

(c) Find $P(X + Y \le 1)$.

The region D where $X+Y \leq 1$ consists of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Integrating our joint density function over this region, we have

$$\begin{aligned} P(X+Y \leq 1) &= \int_0^1 \int_0^{1-x} \frac{1}{2}x(1+y)dydx \\ &= \int_0^1 \frac{1}{2}x\left(\frac{(1+y)^2}{2}\Big|_0^{1-x}\right)dx \\ &= \int_0^1 \frac{1}{2}x\left(\frac{(2-x)^2-1}{2}\right)dx \\ &= \frac{1}{4}\int_0^1 (3x-4x^2+x^3)dx \\ &= \frac{1}{4}\left(\frac{3x^2}{2}-\frac{4x^3}{3}+\frac{x^4}{4}\right)\Big|_0^1 \\ &= \boxed{\frac{5}{48}}. \end{aligned}$$

(9) (a) For the joint density function of the random variable $Z = X+Y$, we note that if we know $Z = z$ and $X = x$, then we must have that $Y = z - x$. So we have the density function $g(z) = \int_{-\infty}^{\infty} f(x, z-x)dx$.

(b) If we know $X = c$, our new joint density function will be

$$g(y) = \frac{f(c,y)}{\int_{-\infty}^{\infty} f(c,y)},$$

where the denominator appears since we need $g(y)$ to be a joint density function.

(c) One can use the same ideas used to find the formulas above to compute these, but one interesting takeaway is that the PDFs for X and Y Pareto random variables show that by far the most probable situations are that one of X and Y is close to 3 and the other is close to 1, while for the normal distribution it is comparatively more likely for both to be closer to 2. If the mean of the normal distribution were 2, this effect would be much more pronounced (and this is the way the problem should originally have been worded, probably).

3. Notes

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked (*) are challenge problems, with problems marked (**) especially challenging problems.