MATH 53 DISCUSSION SECTION ANSWERS – 2/9/23

1. Vector-valued functions: geometric aspects

(1) Many solutions are possible. For example, the curve lies on the surfaces defined by $y = e^{x/2}$, $z = e^x$, and $z = y^2$. This should help sketch the curve (which I won't do here): the curve's projections onto the xy-, xz-, and yz-planes look respectively like two exponential functions and a parabola.

(2) Given that $z = \sqrt{x^2 + y^2} = 1 + y$, we have

$$z^2 = x^2 + y^2 = (1 + y)^2 = 1 + 2y + y^2,$$

so $x^2 = 1+2y$ and therefore $y = \frac{x^2-1}{2}$. Thus we can parametrize the curve by taking $x = t$, $y = \frac{t^2-1}{2}$, and $z = 1 + y = \frac{t^2+1}{2}$. In other words, the curve is given by the vector function

$$\mathrm{r}(t) = \langle t, \frac{t^2 - 1}{2}, \frac{t^2 + 1}{2} \rangle.$$

2. Arc length and curvature

(3) (a) True: parametrizing a curve by arc length means that $|\mathbf{r}'(s)| = 1$.
(b) False: curvature has geometric meaning (i.e. how quickly a curve changes direction) only when it's defined in terms of arc length. If we gave the same definition but with dt in the denominator (for some non-arc length parametrization), then the curvature would depend on our choice of parametrization.

(4) Since this is arc length from $t = 0$ to $t = 1$, this is given by the integral:

$$\begin{aligned}\int_0^1 |\mathbf{r}'(t)|\, dt &= \int_0^1 \left|\langle 0, 2t, 3t^2\rangle\right| dt \\ &= \int_0^1 \sqrt{4t^2 + 9t^4} dt \\ &= \int_0^1 t\sqrt{4 + 9t^2} dt.\end{aligned}$$

To integrate this, we use the substitution $u = 4 + 9t^2$, $du = 18tdt$:

$$\begin{aligned}&= \int_4^{13} \sqrt{u}\frac{du}{18} \\ &= \frac{1}{18}\left[\frac{2u^{3/2}}{3}\right]_4^{13} \\ &= \frac{1}{18}\cdot\frac{2}{3}\left(13^{3/2} - 4^{3/2}\right) \\ &= \frac{13\sqrt{13} - 8}{27} \approx 1.44.\end{aligned}$$

As a sanity check, the straight-line distance between the points $(1, 0, 0)$ and $(1, 1, 1)$ is $\sqrt{2} \approx 1.414$, and it makes sense for the distance along the curve to be slightly larger than this.

(5) The derivative of $\mathbf{r}(t)$ is $\langle 3, 4, 12\rangle$ at all t, and this has magnitude $\sqrt{3^2 + 4^2 + 12^2} = 13$. So the arc length function (starting at the origin) is $s(t) = \int_0^t 13dt = 13t$, and the curve will be parametrized by arc length if we replace t by $s/13$:

$$\mathbf{r}(s) = \langle \frac{3s}{13}, \frac{4s}{13}, \frac{12s}{13} \rangle.$$

(6) We first calculate the derivative and its magnitude:

$$\begin{aligned}\mathbf{r}'(t) &= \langle 1, -3\sin t, 3\cos t\rangle, \text{ so}\\ |\mathbf{r}'(t)| &= \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2}\\ &= \sqrt{1 + 9(\sin^2 t + \cos^2 t)} = \sqrt{10}.\end{aligned}$$

Therefore the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\sqrt{10}} = \langle \frac{1}{\sqrt{10}}, \frac{-3\sin t}{\sqrt{10}}, \frac{3\cos t}{\sqrt{10}}\rangle,$$

and the curvature is

$$\begin{aligned}\kappa(t) &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}\\ &= \frac{\left|\langle 0, \frac{-3\cos t}{\sqrt{10}}, \frac{-3\sin t}{\sqrt{10}}\rangle\right|}{\sqrt{10}}\\ &= \frac{|\langle 0, -3\cos t, -3\sin t\rangle|}{\sqrt{10}\cdot\sqrt{10}}\\ &= \frac{3}{10}.\end{aligned}$$