MATH 53 DISCUSSION SECTION ANSWERS – 2/7/23

1. Lines and planes in 3D space

(1) (a) False. For example, the lines $x = y = 0$ and $y = z = 1$ are skew lines; they are perpendicular but don't intersect.
(b) False. For example, the planes $x = 0$, $y = 0$, and $x + y = 0$ intersect in the line $x = y = 0$.

(2) One normal vector to this plane is $\langle 3, 2, 6\rangle$, and in fact the plane consists of all vectors $\mathbf{v}$ such that $\mathbf{v} \cdot \langle 3, 2, 6\rangle = 5$. The nearest point on the plane to $(1, -2, 4)$ will be the point where the plane intersects the line $\langle 1, -2, 4\rangle + c\langle 3, 2, 6\rangle$. So let's solve for c:

$$\begin{aligned} 5 &= (\langle 1, -2, 4\rangle + c\langle 3, 2, 6\rangle) \cdot \langle 3, 2, 6\rangle \\ &= \langle 1, -2, 4\rangle \cdot \langle 3, 2, 6\rangle + c\left(\langle 3, 2, 6\rangle \cdot \langle 3, 2, 6\rangle\right) \\ &= 23 + 49c, \end{aligned}$$

so $c = \frac{5-23}{49} = \frac{-18}{49}$. Then the distance from the given point to the plane is the magnitude of the offset vector we used:

$$\left|\frac{-18}{49}\langle 3, 2, 6\rangle\right| = \frac{18}{49}\left|\langle 3, 2, 6\rangle\right| = \frac{18}{49} \cdot 7 = \frac{18}{7}.$$

Note: the solution above basically re-derives the formula you'll see in your textbook,

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}},$$

which in this case evaluates to

$$D = \frac{|3 \cdot 1 + 2 \cdot (-2) + 6 \cdot 4 - 5|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{18}{\sqrt{49}} = \frac{18}{7}.$$

(3) One parametric equation is

$$\begin{aligned} \mathbf{r}(t) &= \langle -8, 1, 4\rangle + t\left(\langle 3, -2, 4\rangle - \langle -8, 1, 4\rangle\right) \\ &= \langle -8, 1, 4\rangle + t\left(\langle 11, -3, 0\rangle\right). \end{aligned}$$

We can't use the usual symmetric form because of the 0 in the direction vector (which would appear in a denominator); instead we can say that the line is defined by

$$\frac{x - (-8)}{11} = \frac{y - 1}{-3} \text{ and } z = 4,$$

where the first two expressions are found by solving for and eliminating t.

(4) Any plane parallel to $x + y + z = 0$ has the form $x + y + z = d$, and the only such plane passing through the given point is $x + y + c = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

(5) First let's translate the given points so that one of them (say the first) is at the origin. Then we want to find the plane through the origin containing the vectors $\mathbf{v} = \langle 1, -9, 4\rangle$ and $\mathbf{w} = \langle -4, -4, -1\rangle$. A normal vector to this plane is given by

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -9 & 4 \\ -4 & -4 & -1 \end{vmatrix} \\ &= 25\mathbf{i} - 15\mathbf{j} - 40\mathbf{k}, \end{aligned}$$

so the plane can be described by the equation $25x - 15y - 40z = 0$. To get our final answer, we translate this plane back by $\langle 2, 1, 2\rangle$ to get the equation

$$\begin{aligned} 25(x - 2) - 15(y - 1) - 40(z - 2) = 0, \text{ or} \\ 25x - 15y - 40z = 45. \end{aligned}$$

(You can divide through by 5 if you want to use smaller numbers.)

(6) Plug the parametric equation for the line into the equation for the plane:

$$3(t-1)-(1+2t)+2(3-t)=5 \iff -t+2=5 \iff t=-3.$$

So the point of intersection appears when $t=-3$, yielding the point $(-4,-5,6)$.

(7) The angle between the planes is the angle between their normal vectors, $\mathbf{v}=\langle 1,4,-3\rangle$ and $\mathbf{w}=\langle -3,6,7\rangle$. This can be computed from their dot product:

$$\cos\theta=\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}=\frac{-3+24-21}{\sqrt{26}\sqrt{94}}=0,$$

so $\theta=\pi/2$; that is, the vectors (and thus the planes) are perpendicular.

(8) In general, an m-dimensional plane in n-dimensional space should be described by $n-m$ linear equations, provided that these equations are neither redundant nor contradictory. You can also give an analogue of a parametric equation, but this will require m parameters; e.g. a 2-dimensional plane can be written as

$$\mathrm{v}+\mathrm{w}t+\mathrm{x}u$$

where w and x are linearly independent vectors and $t,u\in\mathrm{R}$ are parameters.

2. Vector-valued functions and space curves

(9) (a) False. Its domain is the intersection of the three domains.
(b) True. We define the angle between the two curves to be the angle between their tangent vectors (which makes sense because these vectors are nonzero).
(c) True. In the context of particle motion, t represents time, so the specific parametrization we're using tells us how quickly the particle traces out a given path (and not just where it goes).

(10) Just differentiate each component: $\langle \frac{1}{2}(t-2)^{-1/2},0,-2t^{-3}\rangle$.

(11) We first calculate the derivative:

$$\mathbf{r}'(t)=\langle 2t-2,3,t^2+t\rangle,$$

which equals $\langle 2,3,6\rangle$ at $t=2$. To find the unit tangent vector, divide this by its magnitude (namely $\sqrt{2^2+3^2+6^2}=\sqrt{49}=7$) to get $\langle \frac{2}{7},\frac{3}{7},\frac{6}{7}\rangle$.

(12) We can just integrate the three components separately. The **i**-component is

$$\int_0^1 \frac{dt}{t+1}=[\ln(t+1)]_0^1=\ln 2-\ln 1=\ln 2.$$

The **j**-component is

$$\int_0^1 \frac{dt}{t^2+1}=\left[\tan^{-1}(t)\right]_0^1=\frac{\pi}{4}-0=\frac{\pi}{4}.$$

To find the **k**-component, substitute $u=t^2+1$, with $du=2tdt$:

$$\int_0^1 \frac{tdt}{t^2+1}=\int_1^2 \frac{du/2}{u}$$
$$=\frac{1}{2}\left[\ln(u)\right]_1^2=\frac{\ln 2-\ln 1}{2}=\frac{\ln 2}{2}.$$

So the answer is

$$(\ln 2)\mathbf{i}+\frac{\pi}{4}\mathbf{j}+\frac{\ln 2}{2}\mathbf{k}.$$