# MATH 53 DISCUSSION SECTION PROBLEMS – 2/28/23

## 1. The multivariable chain rule

(1) True/false practice:
   (a) For a function $f(x, y)$ sufficiently nice to satisfy the hypotheses of Clairaut's theorem and differentiable functions $x(t), y(t)$, we can write $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ or as $\frac{\partial f}{\partial y}\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{dy}{dt}$ since we can interchange the order of differentiation as much as we like.

(2) **(textbook 14.5.9** If $z = \ln(3x + 2y)$, $x = s \sin t$, $y = t \cos s$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(3) **(textbook 14.5.17)** Write the chain rule for the situation where $u = f(x, y)$, $x = x(r, s, t)$, $y = y(r, s, t)$, with $f, x, y$ all differentiable functions.

(4) **(textbook 14.5.33)** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the surface $e^z = xyz$.

(5) **((*), textbook 14.5.53)** If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}.$$

(6) **((**); day 2 of Math 222A)** Using the problem above, find a solution to Laplace's equation $u_{xx} + u_{yy} = 0$ for $u$ a function from $\mathbb{R}^2 \backslash \{(0, 0)\}$ to $\mathbb{R}$ that is radially symmetric (i.e. depends only on the distance from the origin, not the angle from the positive $x$-axis).

## 2. Directional derivatives and the gradient vector

(7) True/false practice:
   (a) Suppose $f$ is a differentiable function of two or more variables. The *minimum* value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ occurs when $\mathbf{u}$ has the opposite direction as the gradient vector $\nabla f(\mathbf{x})$.
   (b) If you look at a contour map of a differentiable function $f(x, y)$ of two variables and take a point $P = (a, b)$ on some level curve $f(x, y) = k$, then $\nabla f(a, b)$ will be parallel to the level curve.

(8) **(textbook 14.6.13)** Find the directional derivative of $g(s, t) = s\sqrt{t}$ at the point $(2, 4)$ in the direction of $2\mathbf{i} - \mathbf{j}$.

(9) **(based on an old quiz question)** Find the directional derivative of $f(x, y, z) = x^2 + y^3 + z^4$ in the direction along the curve $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$.

(10) **(textbook 14.6.23)** Find the maximum rate of change of $f(x, y) = \sin(xy)$ at the point $(1, 0)$ and the direction in which it occurs.

(11) **(textbook 14.6.43)** Find the equation of the tangent plane to the surface $xy^2z^3 = 8$ at the point $(2, 2, 1)$.

(12) **(a bit more abstract)** Let $\mathbf{u} = \langle a, b \rangle$, and let $f(x, y)$ be a twice-differentiable function of two variables. What is $D_{\mathbf{u}}\big(D_{\mathbf{u}} f(x, y)\big)$, the second directional derivative of $f$ in the direction of $\mathbf{u}$?

## 3. Notes

Original author: James Rowan.

All problems labeled "textbook" come from Stewart, James, *Multivariable Calculus: Math 53 at UC Berkeley*, 8th Edition, Cengage Learning, 2016.

Problems marked **(*)** are challenge problems, with problems marked **(**)** especially challenging problems.