MATH 53 DISCUSSION SECTION ANSWERS – 2/28/23

1. The multivariable chain rule

(1) (a) False: that only works when iterating partial differentiation, not when multiplying one derivative by another. Only the first formula is correct.[1]

(2) The multivariable chain rule gives:

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= \frac{3}{3x+2y}\sin t + \frac{2}{3x+2y}\left(-t\sin s\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= \frac{3}{3x+2y}\left(s\cos t\right) + \frac{2}{3x+2y}\cos s.\end{aligned}$$

(Both of these can be expressed in terms of only s and t by plugging in the equations $x = s\sin t$, $y = t\cos s$.)

(3) The formulas are

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \\ \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}, \text{ and} \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.\end{aligned}$$

(4) We use implicit differentiation. Let $F(x, y, z) = e^z - xyz$, which equals zero on the given surface. Then we have

$$\begin{aligned}0 &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= F_x + F_z\frac{\partial z}{\partial x} \\ &= -yz + (e^z - xy)\frac{\partial z}{\partial x},\end{aligned}$$

so

$$\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}.$$

A similar calculation gives

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}.$$

Note that these answers could be obtained more directly using the formulas from the end of section 14.5 in the textbook:

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{\partial F/\partial x}{\partial F/\partial z}, \\ \frac{\partial z}{\partial y} &= -\frac{\partial F/\partial y}{\partial F/\partial z}.\end{aligned}$$

[1] An earlier version of this worksheet said these were formulas for $\frac{df}{dx}$ instead of $\frac{df}{dt}$, in which case both formulas are wrong.

(5) Let's expand the partial derivatives with respect to r and θ using the chain rule:

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\\ &= f_x \cos\theta + f_y \sin\theta \text{ and}\\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\\ &= -f_x r\sin\theta + f_y r\cos\theta.\end{aligned}$$

For the second partial derivatives, we apply the chain rule (and the product rule as necessary) to the input functions $\partial z/\partial r$ and $\partial z/\partial\theta$, which we calculated above:

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(f_x\cos\theta + f_y\sin\theta\right)\\ &= (f_x)_r\cos\theta + (f_y)_r\sin\theta\\ &= (f_{xx}\cos\theta + f_{xy}\sin\theta)\cos\theta + (f_{yx}\cos\theta + f_{yy}\sin\theta)\sin\theta\\ &= f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta, \text{ and}\\ \frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial\theta}\left(-f_x r\sin\theta + f_y r\cos\theta\right)\\ &= \left(-f_x r\cos\theta + \frac{\partial f_x}{\partial\theta} r(-\sin\theta)\right) + \left(f_y r(-\sin\theta) + \frac{\partial f_y}{\partial \theta} r\cos\theta\right)\\ &= \left(-f_x r\cos\theta + \left(f_{xx}\frac{\partial x}{\partial\theta} + f_{xy}\frac{\partial y}{\partial\theta}\right) r(-\sin\theta)\right)\\ &\quad + \left(f_y r(-\sin\theta) + \left(f_{yx}\frac{\partial x}{\partial\theta} + f_{yy}\frac{\partial y}{\partial\theta}\right) r\cos\theta\right)\\ &= \left(-f_x r\cos\theta + (f_{xx}(-r\sin\theta) + f_{xy}(r\cos\theta))\, r(-\sin\theta)\right)\\ &\quad + \left(f_y r(-\sin\theta) + (f_{yx}(-r\sin\theta) + f_{yy}(r\cos\theta))\, r\cos\theta\right)\\ &= -f_x r\cos\theta - f_y r\sin\theta + f_{xx}r^2\sin^2\theta - 2f_{xy}r^2\sin\theta\cos\theta + f_{yy}r^2\cos^2\theta.\end{aligned}$$

Finally, we put everything together to calculate the right-hand side of the equation in question:

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} &= \left(f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta\right) + \frac{1}{r}\left(f_x\cos\theta + f_y\sin\theta\right)\\ &\quad + \frac{1}{r^2}\left(-f_x r\cos\theta - f_y r\sin\theta + f_{xx}r^2\sin^2\theta - 2f_{xy}r^2\sin\theta\cos\theta + f_{yy}r^2\cos^2\theta\right).\end{aligned}$$

The terms involving f_x, f_y, and f_{xy} here cancel out, and the terms involving f_{xx} and f_{yy} combine to make:

$$\begin{aligned}&= f_{xx}\left(\cos^2\theta + \sin^2\theta\right) + f_{yy}\left(\sin^2\theta + \cos^2\theta\right)\\ &= f_{xx} + f_{yy},\end{aligned}$$

as desired.

(6) Suppose $u(x, y)$ is a solution to Laplace's equation that can be written as a single-variable function of the radius, $u(x, y) = g(r)$. Then the previous problem tells us that

$$\begin{aligned}0 = u_{xx} + u_{yy} &= u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r\\ &= g''(r) + \frac{1}{r^2}(0) + \frac{1}{r}g'(r)\\ &= g''(r) + \frac{g'(r)}{r}.\end{aligned}$$

This is a second-order ordinary differential equation, but we can interpret it as a first-order ODE involving the function $h(r) = g'(r)$, namely

$$h'(r) + \frac{h(r)}{r} = 0.$$

To solve this, let's temporarily use the notation $x = r$, $y = h(r)$, and solve by separating variables:

$$\begin{aligned} y' + \frac{y}{x} &= 0 \\ y' &= -\frac{y}{x} \\ \frac{y'}{y} &= -\frac{1}{x} \\ \int \frac{y'}{y} dx &= \int -\frac{1}{x} dx \\ \int \frac{dy}{y} &= \int -\frac{dx}{x} \\ \log y &= -\log x + C \\ y &= e^{-\log x + C} \\ &= \frac{e^C}{x} = \frac{A}{x} \text{ for a constant } A. \end{aligned}$$

So we get $h(r) = A/r$. Integrating with respect to r gives

$$\begin{aligned} g(r) &= \int h(r) dr \\ &= A \log r + C \\ &= A \log\left(\sqrt{x^2 + y^2}\right) + C \\ &= \frac{A}{2} \log(x^2 + y^2) + C. \end{aligned}$$

This is the general radially symmetric solution to Laplace's equation. (Of course we can choose A and C however we like. The case $A = 0$ corresponds to constant solutions, which are technically correct but very boring answers to this question.)

2. Directional derivatives and the gradient vector

(7) (a) True, and this minimum value is exactly the negative of the maximum value.
(b) False: it will be perpendicular to the level curve, pointing in the "steepest uphill" direction.

(8) We first calculate the gradient of g:

$$\begin{aligned} \nabla g &= \langle g_s, g_t \rangle \\ &= \langle \sqrt{t}, s/2\sqrt{t} \rangle \\ &= \langle 2, 1/2 \rangle \text{ at } (s,t) = (2,4). \end{aligned}$$

The directional derivative is the dot product of the gradient with the normalized version of the given vector (namely $\mathbf{u} = \frac{\langle 2,-1 \rangle}{|\langle 2,-1 \rangle|} = \frac{\langle 2,-1 \rangle}{\sqrt{5}}$):

$$\begin{aligned} D_{\mathbf{u}} g(2,4) &= \langle 2, 1/2 \rangle \cdot \langle 2/\sqrt{5}, -1/\sqrt{5} \rangle \\ &= (4 - 1/2)/\sqrt{5} = 7/2\sqrt{5}. \end{aligned}$$

(9) We first find the unit tangent vector to this curve: since

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle = \langle 1, 2, 3 \rangle \text{ at } t = 1,$$

we have

$$\mathbf{u}(t) = \frac{\langle 1,2,3 \rangle}{|\langle 1,2,3 \rangle|} = \frac{\langle 1,2,3 \rangle}{\sqrt{14}}.$$

The gradient of f is

$$\begin{aligned}\nabla f &= \langle f_x, f_y, f_z\rangle \\ &= \langle 2x, 3y^2, 4z^4\rangle \\ &= \langle 2,3,4\rangle \text{ at } (1,1,1),\end{aligned}$$

so the directional derivative is

$$D_{\mathbf{u}} f(1,1,1) = \frac{\langle 1,2,3\rangle}{\sqrt{14}} \cdot \langle 2,3,4\rangle = \frac{20}{\sqrt{14}}.$$

(10) The gradient of f is

$$\begin{aligned}\nabla f &= \langle f_x, f_y\rangle \\ &= \langle y\cos(xy), x\cos(xy)\rangle \\ &= \langle 0,1\rangle \text{ at } (1,0).\end{aligned}$$

The maximum rate of change of f at $(1,0)$ is the magnitude of this vector (namely 1), and the direction in which it occurs is the direction in which the vector points (namely $\mathbf{j}$)

(11) The given surface can be viewed as a level surface of the function $F(x,y,z) = xy^2z^3$. In general, tangent planes to level surfaces are orthogonal to the gradient of the function, so we are looking for the plane passing through the given point $(2,2,1)$ and orthogonal to the gradient vector. Let's calculate the gradient:

$$\begin{aligned}\nabla F &= \langle F_x, F_y, F_z\rangle \\ &= \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle \\ &= \langle 4,8,24\rangle \text{ at } (2,2,1).\end{aligned}$$

The plane orthogonal to this and passing through $(2,2,1)$ is given by the equation

$$4(x-2) + 8(y-2) + 24(z-1) = 0.$$

(12) In the special cases $\mathbf{u} = \mathbf{i}$ or $\mathbf{j}$, this is just f_{xx} or f_{yy} respectively. In general, it's the second derivative of the function given by slicing the graph of f with a vertical plane containing the vector $\mathbf{u}$.