MATH 53 DISCUSSION SECTION ANSWERS – 2/23/23

1. Tangent planes

(1) (a) False: either the first two coordinates or the last coordinate of the given vector should have minus signs.

(b) False: just as in the single-variable case, there are plenty of functions that are continuous but not differentiable at a given point, like $f(x,y) = |x|$ at the origin. There are even functions (which are much harder to write down) which are continuous everywhere but differentiable nowhere.

(2) We calculate that $f_x(x,y) = \ln(xy-5) + \frac{xy}{xy-5}$ and $f_y(x,y) = \frac{x^2}{xy-5}$. These functions are both continuous at $(2,3)$ (since $xy - 5 \neq 0$ there), so f is differentiable at $(2,3)$. The linearization is given by

$$\begin{aligned} L(x,y) &= f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) \\ &= (1 + 2\ln 1) + \left(\ln 1 + \frac{6}{1}\right)(x-2) + \left(\frac{4}{1}\right)(y-3) \\ &= 1 + 6(x-2) + 4(y-3). \end{aligned}$$

(3) We approximate the function with its linearization:

$$\begin{aligned} f(2.2, 4.9) \approx L(2.2, 4.9) &= 6 + 1(2.2-2) - 1(4.9-5) \\ &= 6 + 0.2 + 0.1 = 6.3. \end{aligned}$$

We could make a similar approximation at the point $f(3,6)$, but it would be a less reliable approximation since $(3,6)$ is farther away from $(2,5)$.

(4) The volume of a cylinder with diameter x and height y is $f(x,y) = \pi(x/2)^2 y$. The tin in the can consists of a cylinder with a smaller inner cylinder removed from it, so its volume is $f(8,12) - f(7.92, 11.92)$. To approximate $f(7.92, 11.92)$, we note that $f_x = \frac{\pi xy}{2}$ and $f_y = \pi(x/2)^2$, so the linear approximation gives

$$\begin{aligned} f(7.92, 11.92) \approx L(7.92, 11.92) &= f(8,12) + f_x(8,12)(7.92-8) + f_y(8,12)(11.92-12) \\ &= f(8,12) + 48\pi(-0.08) + 16\pi(-0.08) \\ &= f(8,12) - 5.12\pi. \end{aligned}$$

So the volume of the tin is approximately 5.12π cubic centimeters.

(5) (a) Rather than taking partial derivatives of the function $z = \sqrt[4]{18 - x^4 - y^4}$, let's use a shortcut. From the equation $x^4 + y^4 + z^4 = 18$, take differentials of both sides:

$$\begin{aligned} d(x^4 + y^4 + z^4) &= d(18) \\ 4x^3dx + 4y^3dy + 4z^3dz &= 0 \\ dz &= -\frac{4x^3dx + 4y^3dy}{4z^3}. \end{aligned}$$

At the point $(2,1,1)$, this says

$$dz = -\frac{32dx + 4dy}{4} = -8dx - dy.$$

So the tangent plane is given by

$$\begin{aligned} z - 1 &= -8(x-2) - (y-1), \text{ or} \\ z &= -8x - y + 18. \end{aligned}$$

(b) Plugging in $x = 1.99$ and $y = 1.01$, the linear approximation says that

$$z \approx -8(1.99) - 1.01 + 18 = 1.07.$$

We expect this to be a good approximation because $(1.99, 1.01)$ is very close to $(2, 1)$.

(6) Expand the function $x^2 + y^2$ to separate out terms involving $\Delta x = (x - a)$ and $\Delta y = (y - b)$:

$$\begin{aligned} x^2 + y^2 &= ((x-a)+a))^2 + ((y-b)+b)^2 \\ &= a^2 + b^2 + 2a(x-a) + 2b(y-b) + (x-a)^2 + (y-b)^2. \end{aligned}$$

Then $\Delta f = (x^2 + y^2) - (a^2 + b^2)$ has the form

$$\Delta f = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $f_x(a,b) = 2a$, $f_y(a,b) = 2b$, $\epsilon_1 = \Delta x = x - a$, and $\epsilon_2 = \Delta y = y - b$. Note that ϵ_1 goes to 0 as $x \to a$ and ϵ_2 goes to 0 as $y \to b$.

(7) We know from geometry that the tangent plane to any point on a sphere is perpendicular to the axis passing through that point. For the point (x_0, y_0, z_0), any plane perpendicular to this has the form $x_0x + y_0y + z + 0z = d$. The only plane of this form passing through (x_0, y_0, z_0) is the one with $d = \langle x_0, y_0, z_0 \rangle \cdot \langle x_0, y_0, z_0 \rangle = 1$.